Simple proof of the completeness theorem for second order classical and intuitionistic logic

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Abstract

We present a simpler way than usual to deduce the completeness theorem for the second-oder classical logic from the first-order one. We also extend our method to the case of second-order intuitionnistic logic.

1 Introduction

The usual way (but not the original Henkin's proof [1, 2]) for proving the completeness theorem for second-order logic is to deduce it from the completeness theorem for first-order multi-sorted logic. There is clearly a trivial translation from second-order logic to first-order multi-sorted logic, by associating one sort to first-order object and, for each $n \in \mathbb{N}$, one sort for predicate of arity n.
Another way is to deduce it from the completeness theorem for first-order mono-sorted logic (one may find it in [6]). To do this, a first-order variable x is associated to each second-order variable X of ariry n, and the atomic formula $X(x_1, \ldots, x_n)$ is encoded by $\mathrm{Ap}_n(x, x_1, \ldots, x_n)$ where Ap_n is a relation symbol of arity $n+1$. Then, this coding is extended to all formulas.We write it $F \mapsto F^*$. However, to allow the translation between second-order proofs and first-order proofs, one adds some axioms to discriminate between first and second-order objects. The critical point is the translation of quantifications:

- For first-order quantification we define $(\forall x\, F)^* = \forall x(v(x) \to F^*)$ where v is a new predicate constant.
- For second-order quantification of arity n we define $(\forall X^n\, F)^* = \forall x(V_n(x) \to F^*)$ where V_n is a new predicate constant.

Then we add axioms relating v, V_n and Ap_n such as $\forall x, y(\mathrm{Ap}_1(x, y) \to V_1(x) \wedge v(y))$.
The problem is that this translation is not surjective. So it is not immediate to prove that if F^* is provable in first-order logic then F is provable in second-order logic, because all the

formulas appearing in the proof of F^* are not necessarely of the shape G^*. It is not even clear that the proof in [6] which is only sketched can be completed into a correct proof (at least the authors do not know how to end his proof). May be there is a solution using the fact that subformulas of F^* are nearly of the shape G^* and one could use this in a direct, but very tedious, proof by induction on the proof of F using the subformula property which is a strong result.
Our solution, is to use no axiom nor extra predicate constant, but to give two codings $F \mapsto F^*$ from second-order logic to first-order and $F \mapsto F^\diamond$ from first-order logic to second-order such that $F^{*\diamond}$ and F are logicaly equivalent. To achieve this we consider that in first order logic the same variable may have different meanings (in the semantics) depending on it's position in atomic formulas. Thus, we can translate any first-order formula back to a second-order formula.
Using this method we can also deduce easily a completeness theorem for second-order intuitionnistic logic. This was not at all so clear with the original method (it is not clear if we need classical absurdity to use the extra axioms). We also give some simple examples showing that despite a complex definition, computation are possible in these kind of models.

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2 Coding

Definition 2.1 (second-order language) *Let $\mathcal{L}_2$, the language of second-order logic, be the following:*

- *The logical symbols $\bot, \to$, $\wedge$, $\vee$, $\forall$ and $\exists$.*
- *A countable set $\mathcal{V}$ of first-order variables : $x_0, x_1, x_2, \ldots$*
- *A countable set Σ of constants and functions symbols (of various arity) : $a, b, f, g, h, \ldots$.*
- *Using $\mathcal{V}$ and Σ we construct the set of first-order terms $\mathcal{T}$: $t_1, t_2, \ldots$*
- *For each $n \in \mathbb{N}$, a countable set $\mathcal{V}_n$ of second-order variables of arity n : $X_0^n, X_1^n, X_2^n, \ldots$.*

To simplify, we omit second-order constants (they can be replaced by free variables).

Definition 2.2 (first-order language) *Let $\mathcal{L}_1$, a particular langage of first-order logic, be the following:*

- *The logical symbols $\bot, \to$, $\wedge$, $\vee$, $\forall$ and $\exists$.*
- *A countable set $\mathcal{V}$ of first-order variables : $x_0, x_1, x_2, \ldots$ (it is simpler to use the same set of first-order variables in $\mathcal{L}_1$ and $\mathcal{L}_2$).*
- *A countable set Σ of constants and functions symbols (of various arity) : $a, b, f, g, h, \ldots$. Here again we use the same set than for $\mathcal{L}_2$.*
- *For each $n \in \mathbb{N}$, a relation symbol Ap_n of arity $n+1$.*

Notations

- We write $\mathcal{F}_v(F)$ for the set of all free variables of a formula F.
- We write $F \leftrightarrow G$ for $(F \to G) \wedge (G \to F)$.
- We write $F[x := t]$ for the first-order substitution of a term.
- We write $F[X^n := Y^n]$ for the second-order substitution of a variable.
- We write $F[X^n := \lambda x_1 \dots x_n G]$ for the second-order substitution of a formula.
- We will use natural deduction [5, 6] both for second and first-order logic, and we will write $\Gamma \vdash_k^n F$ with $k \in \{i, c\}$ (for intuitionistic or classical logic) and $n \in \{1, 2\}$ (for first or second-order).

We have the following lemma:

Lemma 2.3 *If $\Gamma \vdash_k^n A$ then, for every substitution σ, $\Gamma[\sigma] \vdash_k^n A[\sigma]$.*

Definition 2.4 (coding) *We choose for each $n \in \mathbb{N}$ an isomorphism ϕ_n from $\mathcal{V}_n$ to $\mathcal{V}$. The fact that it is an isomorphism for each n is the main point in our method.*
Let F be a second-order formula, we define a first-order formula F^* by induction as follows:

- $\bot^* = \bot$
- $(X^n(t_1, \dots, t_n))^* = Ap_n(\phi_n(X^n), t_1, \dots, t_n)$
- $(A \Diamond B)^* = A^* \Diamond B^*$ *where* $\Diamond \in \{\to, \wedge, \vee\}$
- $(Qx\, A)^* = Qy(A[x := y])^*$ *where* $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$
- $(QX^n\, A)^* = Qy(A[X^n := Y^n])^*$ *where* $\Phi_n(Y^n) = y$, $y \notin \mathcal{F}_v(A^*)$ *and* $Q \in \{\forall, \exists\}$

Example 2.5 $(\forall X(X(x) \to X(y)))^* = \forall z(Ap_1(z, x) \to Ap_1(z, y))$. *This example illustrates why we need renaming: $\Phi_1(X)$ may be already used in $(X(x) \to X(y))^*$.*

Remark 2.6 *The mapping $F \mapsto F^*$ is not surjective, for instance there is no antecedent for $\forall x\, Ap_1(x, x)$ or $Ap_1(f(a), a)$.*

Definition 2.7 (comprehension schemas) *The second-order comprehension schema SC_2 is the set of all closed formulas $SC_2(G; x_1, \dots, x_n; \chi_1, \dots, \chi_m)$ where $\{x_1, \dots, x_n\} \subset \mathcal{V}$ and $\mathcal{F}_v(G) \subseteq \{x_1, \dots, x_n, \chi_1, \dots, \chi_m\}$ and*

$$SC_2(G; x_1, \dots, x_n; \chi_1, \dots, \chi_m) = \forall \chi_1 \dots \forall \chi_m \exists X^n \forall x_1 \dots \forall x_n\, (G \leftrightarrow X^n(x_1, \dots, x_n)) \in SC_2$$

where $X^n \notin F_v(G)$.
The first-order comprehension schema SC_1 is defined simply as $SC_2^* = \{F^*, F \in SC_2\}$

It is easy to show that SC_2 is provable in second order logic.

Remark 2.8 *Let $F = X(x)$ where $\Phi_1(X) = x$. We have:*

- $SC_2(F;x;X) = \forall X \exists Y \forall x (F \leftrightarrow Y(x)) \in SC_2$.
- $SC_2(F;x;X)^* = (\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (Ap_1(z,x) \leftrightarrow Ap_1(y,x)) \in SC_1$.

It is easy to see that $(\forall X \exists Y \forall x (F \leftrightarrow Y(x)))^* = \forall z \exists y \forall x (F[X := Z]^* \leftrightarrow Ap_1(y,x))$ *where* $\phi_1(Z) = z \neq x$.
In general we have the following result : for each second-order formula G *there is a variable substitution* σ *such that*

$$\begin{aligned} SC_2(G;x_1,\ldots,x_n;\chi_1,\ldots,\chi_m)^* &= (\forall \chi_1 \ldots \forall \chi_m \exists X^n \forall x_1 \ldots \forall x_n \, (G \leftrightarrow X^n(x_1,\ldots,x_n)))^* \\ &= \forall y_1 \ldots \forall y_m \exists x \forall x_1 \ldots \forall x_n \, (G[\sigma]^* \leftrightarrow Ap_n(x,x_1,\ldots,x_n)) \, . \end{aligned}$$

We can now show the following theorem (we will not use it):

Theorem 2.9 *Let* Γ *be a second-order context and* A *a second-order formula. If* $\Gamma \vdash_k^2 A$ *then* $\Gamma^*, SC_1 \vdash_k^1 A^*$ $(k \in \{i,c\})$.

proof: By induction on the derivation of $\Gamma \vdash_k^2 A$, using SC_1, remark 2.8 and lemma 2.3 for the case of the second-order elimination of $\forall$ and the second-order introduction of $\exists$. □

Definition 2.10 (reverse coding) *Let* F *be a first-order formula, we define a second-order formula* $F^\diamond$ *by induction as follows:*

- $\bot^\diamond = \bot$
- $Ap_n(x,t_1,\ldots,t_n)^\diamond = X^n(x_1,\ldots,x_n)$ *where* $X^n = \phi_n^{-1}(x)$
- $Ap_n(t,t_1,\ldots,t_n)^\diamond = \bot$ *if* t *is not a variable.*
- $(A \Diamond B)^\diamond = A^\diamond \Diamond B^\diamond$ *where* $\Diamond \in \{\rightarrow, \wedge, \vee\}$
- $(Qx\, A)^\diamond = QxQX^{i_1} \ldots QX^{i_p} A^\diamond$ *where* $Q \in \{\forall, \exists\}$, $X^n = \phi_n^{-1}(x)$ *for all* $n \in \mathbb{N}$, $i_1 < i_2 < \ldots < i_p$ *and* $\{X^{i_1}, \ldots, X^{i_p}\} = \{X^n; n \in \mathbb{N}\} \cap \mathcal{F}_v(A^\diamond)$

Remark 2.11 *We don't need renaming in order to define* $(Qx\, A)^\diamond$ *since the* ϕ_n *are isomorphisms.*

Lemma 2.12 *If* A *is a second order formula then* $\vdash_i^2 A^{*\diamond} \leftrightarrow A$.

proof: By induction on the formula A. □

Remark 2.13 *We can not say that* $A^{*\diamond} = A$, *because in the case of the quantifier, we can add or remove some quantifiers on variables with no occurrence. For instance, if* X^1 *and* x *are not free in* A *and* $\Phi_1(X^1) = x$ *then* $(\forall X\, A)^* = (\forall x\, A)^*$ *and* $(\forall X\, A)^{*\diamond} = \forall x\, A$. □

Corollary 2.14 $\vdash_i^2 (SC_1)^\diamond \leftrightarrow SC_2$ *which means that each formula in* $(SC_1)^\diamond$ *is equivalent to at least one formula in* SC_2 *and vice versa.*

proof: Consequence of 2.12. □

Example 2.15 *Let* $F = \forall x (Ap_1(x,y) \rightarrow Ap_2(x,y,y) \vee Ap_1(y,x))$ *and*
$G = Ap_1(t,y) \rightarrow Ap_2(t,y,y) \vee Ap_1(y,t)$.
We have :

- $F \vdash_i^1 G$
- $F^\diamond = \forall x \forall X^1 \forall X^2 (X^1(y) \to X^2(y,y) \vee Y^1(x))$ *where* $\phi_1(Y^1) = y$
- *If* $t = z$*, then* $G^\diamond = Z^1(y) \to Z^2(y,y) \vee Y^1(z)$ *where* $\phi_1(Z^1) = \phi_2(Z^2) = z$
- *Si* t *is not a variable, then* $G^\diamond = \perp \to \perp \vee Y^1(t)$

We remark that :

- $\{X^1(y) \to X^2(y,y) \vee Y^1(x)\}[X^1 := Z^1][x := z] = Z^1(y) \to Z^2(y,y) \vee Y^1(z)$
- $\{X^1(y) \to X^2(y,y) \vee Y^1(x)\}[X^1 := \lambda x_1 \perp][x := t] = \perp \to \perp \vee Y^1(t)$

and then $F^\diamond \vdash_i^2 G^\diamond$.

Lemma 2.16 *Let* Γ *be a fist-order context and* A *a first-order formula. If* $\Gamma \vdash_k^1 A$ *then* $\Gamma^\diamond \vdash_k^2 A^\diamond$ $(k \in \{i, c\})$.

proof: By induction on the derivation of $\Gamma \vdash_k^1 A$. The only difficult cases are the case of the elimination of $\forall$ and the introduction of $\exists$ which are treated in the same way that the examples 2.15. □

Now, we can prove the converse of theorem 2.9, which is the main tool to prove our completeness theorems:

Theorem 2.17 *Let* Γ *be a second-order context and* A *a second-order formula. If* $\Gamma^*, SC_1 \vdash_k^1 A^*$ *then* $\Gamma \vdash_k^2 A$ $(k \in \{i, c\})$.

proof: By lemma 2.16, corollary 2.14, lemma 2.12 and using the fact that formulas in SC_2 are provable. □

3 Classical completeness

Definition 3.1 (second-order classical model) *A second-order model for* $\mathcal{L}_2$ *is given by a tuple* $\mathcal{M}_2 = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n \in \mathbb{N}})$ *where*

- $\mathcal{D}$ *is a non empty set.*
- $\overline{\Sigma}$ *contains a function* $\overline{f}$ *from* $\mathcal{D}^n$ *to* $\mathcal{D}$ *for each function* f *of arity* n *in* Σ.
- $\mathcal{P}_n \subseteq \mathcal{P}(\mathcal{D}^n)$ *for each* $n \in \mathbb{N}$*. The set* $\mathcal{P}_n$ *of subsets of* $\mathcal{D}^n$ *will be use as range for the second-order quantification of arity* n*. For* $n = 0$*, we assume that* $\mathcal{P}_0 = \mathcal{P}(\mathcal{D}^0) = \{0, 1\}$ *because* $\mathcal{P}(\mathcal{D}^0) = \mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$.

An $\mathcal{M}_2$*-interpretation* σ *is a function on* $\mathcal{V} \cup \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ *such that* $\sigma(x) \in \mathcal{D}$ *for* $x \in \mathcal{V}$ *and* $\sigma(X^n) \in \mathcal{P}_n$ *for* $X^n \in \mathcal{V}_n$.
If σ *is a* $\mathcal{M}_2$*-interpretation, we define* $\sigma(t)$ *the interpretation of a first-order term by induction with* $\sigma(f(t_1, \ldots, t_n)) = \overline{f}(\sigma(t_1), \ldots, \sigma(t_n))$.
Then if σ *is a* $\mathcal{M}_2$*-interpretation we define* $\mathcal{M}_2, \sigma \models A$ *for a formula* A *by induction as follows:*

- *$\mathcal{M}_2, \sigma \models X^n(t_1, \ldots, t_n)$ iff $(\sigma(t_1), \ldots, \sigma(t_n)) \in \sigma(X^n)$*
- *$\mathcal{M}_2, \sigma \models A \to B$ iff $\mathcal{M}_2, \sigma \models A$ implies $\mathcal{M}_2, \sigma \models B$*
- *$\mathcal{M}_2, \sigma \models A \wedge B$ iff $\mathcal{M}_2, \sigma \models A$ and $\mathcal{M}_2, \sigma \models B$*
- *$\mathcal{M}_2, \sigma \models A \vee B$ iff $\mathcal{M}_2, \sigma \models A$ or $\mathcal{M}_2, \sigma \models B$*
- *$\mathcal{M}_2, \sigma \models \forall x\, A$ iff for all $v \in \mathcal{D}$ we have $\mathcal{M}_2, \sigma[x := v] \models A$*
- *$\mathcal{M}_2, \sigma \models \exists x\, A$ iff there exists $v \in \mathcal{D}$ such that $\mathcal{M}_2, \sigma[x := v] \models A$*
- *$\mathcal{M}_2, \sigma \models \forall X^n\, A$ iff for all $\pi \in \mathcal{P}_n$ we have $\mathcal{M}_2, \sigma[X^n := \pi] \models A$*
- *$\mathcal{M}_2, \sigma \models \exists X^n\, A$ iff there exists $\pi \in \mathcal{P}_n$ such that $\mathcal{M}_2, \sigma[X^n := \pi] \models A$*

We will write $\mathcal{M}_2 \models A$ if for all $\mathcal{M}_2$-interpretation σ we have $\mathcal{M}_2, \sigma \models A$.

Definition 3.2 (first-order classical model) *A first-order model for $\mathcal{L}_1$ is given by a tuple $\mathcal{M}_1 = (\mathcal{D}, \overline{\Sigma}, \{\alpha_n\}_{n \in \mathbb{N}})$ where*

- *$\mathcal{D}$ is a non empty set.*
- *$\overline{\Sigma}$ contains a function $\overline{f}$ from $\mathcal{D}^n$ to $\mathcal{D}$ for each function f of arity n in Σ.*
- *$\alpha_n \subseteq \mathcal{D}^{n+1}$ for each $n \in \mathbb{N}$. The relation α_n will be the interpretation of Ap_n.*

An $\mathcal{M}_1$-interpretation σ is a function from $\mathcal{V}$ to $\mathcal{D}$.
For any first-order model $\mathcal{M}_1$, any first-oder formula A and any $\mathcal{M}_1$-interpretation σ, we define $\mathcal{M}_1, \sigma \models A$ et $\mathcal{M}_1 \models A$ as above by induction on A (we just to remove the cases for second-order quantification).

Definition 3.3 (semantical translation) *Let $\mathcal{M}_1 = (\mathcal{D}, \overline{\Sigma}, \{\alpha_n\}_{n \in \mathbb{N}})$ be a first-order model. We define a second-order model ${\mathcal{M}_1}^\diamond = (\mathcal{D}, \overline{\Sigma}, \{\mathcal{P}_n\}_{n \in \mathbb{N}})$ where $\mathcal{P}_0 = \{0, 1\}$ and for $n > 0$, $\mathcal{P}_n = \{|a|_n ; a \in \mathcal{D}\}$ where $|a|_n = \{(a_1, \ldots, a_n) \in \mathcal{D}^n ; (a, a_1, \ldots, a_n) \in \alpha_n\}$.*
Let σ be an $\mathcal{M}_1$-interpretation, we define $\sigma^\diamond$ an ${\mathcal{M}_1}^\diamond -$ interpretation by $\sigma^\diamond(x) = \sigma(x)$ if $x \in \mathcal{V}$ and $\sigma^\diamond(X^n) = |\sigma(\phi(X^n))|_n$.

Lemma 3.4 *For any first-order model $\mathcal{M}_1$, any $\mathcal{M}_1$-interpretation σ and any second order formula A, $\mathcal{M}_1, \sigma \models A^*$ if and only if ${\mathcal{M}_1}^\diamond, \sigma^\diamond \models A$.*

proof: By induction on the formula A. □

Corollary 3.5 *For any first-order model $\mathcal{M}_1$, $\mathcal{M}_1 \models SC_1$ if and only if ${\mathcal{M}_1}^\diamond \models SC_2$.*

proof: Immediate consequence of lemma 3.4 using the fact that formulas in SC_1 and SC_2 are closed. □

Theorem 3.6 (Completeness of second order classical semantic) *Let A be a closed second-order formula. $\vdash^2_c A$ iff for any second-order model $\mathcal{M}_2$ such that $\mathcal{M}_2 \models SC_2$ we have $\mathcal{M}_2 \models A$.*

proof: $\Longrightarrow$ Usual direct proof by induction on the proof of $\vdash^2_c A$.
$\Longleftarrow$ Let $\mathcal{M}_1$ be a first-order model such that $\mathcal{M}_1 \models SC_1$. Using corollary 3.5 we have ${\mathcal{M}_1}^\diamond \models SC_2$ and by hypothesis, we get ${\mathcal{M}_1}^\diamond \models A$. Then using lemma 3.4 we have $\mathcal{M}_1 \models A^*$. As this is true for any first-order model satisfying SC_1, the first-order completeness theorem gives $SC_1 \vdash^1_c A^*$ and this leads to the wanted result $\vdash^2_c A$ using theorem 2.17. □

4 Intuitionnistic completeness

Definition 4.1 (second-order intuitionnistic model) *A second-order Kripke model for* $\mathcal{L}_2$ *is given by a tuple* $\mathcal{K}_2 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}})$ *where*

- $(\mathcal{K}, \leq, 0)$ *is a partially ordered set with* 0 *as bottom element.*
- $\mathcal{D}_p$ *are non empty sets such that for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies* $\mathcal{D}_p \subseteq \mathcal{D}_q$.
- $\overline{\Sigma}_p$ *contains a function* $\overline{f}_p$ *from* $\mathcal{D}_p^n$ *to* $\mathcal{D}_p$ *for each function* f *of arity* n *in* Σ. *Moreover, for all* $p, q \in \mathcal{K}$, $p \leq q$ *implies that for all* $(a_1, \ldots, a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ *we have* $\overline{f}_p(a_1, \ldots, a_n) = \overline{f}_q(a_1, \ldots, a_n)$.
- $\Pi_{n,p}$ *are non empty sets of increasing functions* $(P_q)_{q\geq p}$ *such that for all* $q \geq p, P_q \in \mathcal{P}(\mathcal{D}_q^n)$ *(increasing means for all* $q, q' \geq p$, $q \leq q'$ *implies* $P_q \subseteq P_{q'}$*). Moreover, if* $q \geq p$ *and* $\pi \in \Pi_{n,p}$ *then* π *restricted to all* $q' \geq q$ *belongs to* $\Pi_{n,q}$.

  In particular, an element of $\Pi_{0,p}$ *is a particular increasing function in {0,1} with* $0 = \emptyset$ *and* $1 = \{\emptyset\}$.

A $\mathcal{K}_2$*-interpretation* σ *at level* p *is a function* σ *such that* $\sigma(x) \in \mathcal{D}_p$ *for* $x \in \mathcal{V}$ *and* $\sigma(X^n) \in \Pi_{n,p}$ *for* $X^n \in \mathcal{V}_n$.

Important remark: *if* σ *is a* $\mathcal{K}_2$*-interpretation at level* p *and* $p \leq q$ *then* σ *can be considered as* $\mathcal{K}_2$*-interpretation at level* q *by restricting all the values of second order variables to* $q' \geq q$.

If σ *is a* $\mathcal{K}_2$*-interpretation at level* p*, we define* $\sigma(t)$ *the interpretation of a first-order term by induction with* $\sigma(f(t_1, \ldots, t_n)) = \overline{f}_p(\sigma(t_1), \ldots, \sigma(t_n))$.
Then if σ *is a* $\mathcal{K}_2$*-interpretation at level* p *we define* $\mathcal{K}_2, \sigma, p \Vdash A$ *for a formula* A *by induction as follows:*

- $\mathcal{K}_2, \sigma, p \Vdash X^n(t_1, \ldots, t_n)$ *iff* $(\sigma(t_1), \ldots, \sigma(t_n)) \in \sigma(X^n)(p)$
- $\mathcal{K}_2, \sigma, p \Vdash A \to B$ *iff for all* $q \geq p$ *if* $\mathcal{K}_2, \sigma, q \Vdash A$ *then* $\mathcal{K}_2, \sigma, q \Vdash B$
- $\mathcal{K}_2, \sigma, p \Vdash A \wedge B$ *iff* $\mathcal{K}_2, \sigma, p \Vdash A$ *and* $\mathcal{K}_2, \sigma, p \Vdash B$
- $\mathcal{K}_2, \sigma, p \Vdash A \vee B$ *iff* $\mathcal{K}_2, \sigma, p \Vdash A$ *or* $\mathcal{K}_2, \sigma, p \Vdash B$
- $\mathcal{K}_2, \sigma, p \Vdash \forall x\, A$ *iff for all* $q \geq p$*, for all* $v \in \mathcal{D}_q$ *we have* $\mathcal{K}_2, \sigma[x := v], q \Vdash A$
- $\mathcal{K}_2, \sigma, p \Vdash \exists x\, A$ *iff there exists* $v \in \mathcal{D}_p$ *such that* $\mathcal{K}_2, \sigma[x := v], p \Vdash A$
- $\mathcal{K}_2, \sigma, p \Vdash \forall X^n\, A$ *iff for all* $q \geq p$*, for all* $\pi \in \Pi_{n,q}$ *we have* $\mathcal{K}_2, \sigma[X^n := \pi], q \Vdash A$
- $\mathcal{K}_2, \sigma, p \Vdash \exists X^n\, A$ *iff there exists* $\pi \in \Pi_{n,p}$ *such that* $\mathcal{K}_2, \sigma[X^n := \pi], p \Vdash A$

We will write $\mathcal{K}_2 \Vdash A$ *if for all* $\mathcal{K}_2$*-interpretation* σ *at level* 0 *we have* $\mathcal{K}_2, \sigma, 0 \Vdash A$.

Definition 4.2 (first-order intuitionnistic model) *A first-order Kripke model is given by a tuple* $\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N},p\in\mathcal{K}}, \Vdash)$ *where*

- $(\mathcal{K}, \leq, 0)$ *is a partially ordered set with* 0 *as bottom element.*

- *$\mathcal{D}_p$ are non empty sets such that for all $p, q \in \mathcal{K}$, $p \leq q$ implies $\mathcal{D}_p \subseteq \mathcal{D}_q$.*
- *$\overline{\Sigma}_p$ contains a function $\overline{f}_p$ from $\mathcal{D}_p^n$ to $\mathcal{D}_p$ for each function f of arity n in Σ. Moreover, for all $p, q \in \mathcal{K}$, $p \leq q$ implies that for all $(a_1, \ldots, a_n) \in \mathcal{D}_p^n \subseteq \mathcal{D}_q^n$ we have $\overline{f}_p(a_1, \ldots, a_n) = \overline{f}_q(a_1, \ldots, a_n)$.*
- *$\alpha_{n,p}$ are subsets of $\mathcal{D}_p^{n+1}$ such that for all $p, q \in \mathcal{K}$, for all $n \in \mathbb{N}$, $p \leq q$ implies $\alpha_{n,p} \subseteq \alpha_{n,q}$.*
- *$\Vdash$ is the relation defined by $p \Vdash Ap_n(a, a_1, \ldots, a_n)$ if and only if $p \in \mathcal{K}$ and $(a, a_1, \ldots, a_n) \in \alpha_{n,p}$.*

A $\mathcal{K}_1$-interpretation σ at level p is a function from $\mathcal{V}$ to $\mathcal{D}_p$.
For any first-order Kripke model $\mathcal{K}_1$, any first-oder formula A and any $\mathcal{K}_1$-interpretation σ, we define $\mathcal{K}_1, p, \sigma \Vdash A$ as above.
We will write $\mathcal{K}_1 \Vdash A$ iff for $\mathcal{K}_1$-interpretation σ at level 0 we have $\mathcal{K}_1, \sigma, 0 \Vdash A$.

Definition 4.3 (semantical translation) *Let*

$$\mathcal{K}_1 = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\alpha_{n,p}\}_{n\in\mathbb{N}, p\in\mathcal{K}}, \Vdash)$$

be a first-order Kripke model. We define a second-order Kripke model

$$\mathcal{K}_1{}^{\diamond} = (\mathcal{K}, 0, \leq, \{\mathcal{D}_p\}_{p\in\mathcal{K}}, \{\overline{\Sigma}_p\}_{p\in\mathcal{K}}, \{\Pi_{n,p}\}_{n\in\mathbb{N}, p\in\mathcal{K}})$$

where $\Pi_{n,p} = \{|a|_n; a \in \mathcal{D}_p\}$ with for all $q \geq p$, $|a|_n(q) = \{(a_1, \ldots, a_n) \in \mathcal{D}_q^n; (a, a_1, \ldots, a_n) \in \alpha_{n,q}\}$.
Let σ be a $\mathcal{K}_1$-interpretation at level p, we define $\sigma^{\diamond}$ a $\mathcal{K}_1{}^{\diamond}$-interpretation at level p by $\sigma^{\diamond}(x) = \sigma(x)$ and $\sigma^{\diamond}(X^n) = |\sigma(\phi(X^n))|_n$.

Lemma 4.4 *For any first-order Kripke model $\mathcal{K}_1$, any $\mathcal{K}_1$-interpretation σ at level p and any second order formula A, $\mathcal{K}_1, \sigma, p \Vdash A^*$ if and only if $\mathcal{K}_1{}^{\diamond}, \sigma^{\diamond}, p \Vdash A$.*

proof: By induction on the formula A. □

Corollary 4.5 *For any first-order Kripke model $\mathcal{K}_1$, $\mathcal{K}_1 \Vdash SC_1$ if and only if $\mathcal{K}_1{}^{\diamond} \Vdash SC_2$.*

proof: Immediate consequence of lemma 4.4. □

Theorem 4.6 (Completeness of second order intuitionnistic semantic) *Let A be a closed second-order. $\vdash_i^2 A$ iff for all second-order Kripke model $\mathcal{K}_2$ such that $\mathcal{K}_2 \Vdash SC_2$ we have $\mathcal{K}_2 \Vdash A$.*

proof: $\Longrightarrow$ Usual direct proof by induction on the proof of $\vdash_i^2 A$.
$\Longleftarrow$ Identical to the proof of theorem 3.6 using the lemmas 4.4 and 4.5 instead of lemmas 3.4 and 3.5. □

5 Examples of second order intuitionnistic models

We will now construct a counter model for the universally quantifier peirce's law: $P = \forall X \forall Y(((X \to Y) \to X) \to X)$: We take a model $\mathcal{K}_2$ with to points $0, p$ and such that $\Pi_{0,0}$ contains π_1 and π_2 defined by $\pi_1(0) = \pi_2(0) = \pi_2(p) = 0$ and $\pi_1(p) = 1$. It is clear that $\mathcal{K}_2, \sigma[X := \pi_1, Y := \pi_2], 0 \not\Vdash ((X \to Y) \to X) \to X$. So we have $\mathcal{K}_2 \not\Vdash P$.

Remark 5.1 *The interpretation of a propositionnal variable at level* p *can be seen as a bar: a bar being a set* $\mathcal{B}$ *with*

- *for all* $q \in \mathcal{B}$, $q \geq p$
- *for all* $q, q' \in \mathcal{B}$, *we have neither* $q \leq q'$ *nor* $q' \leq q$

There is a canonical isomorphism between the set of bar and the set of increasing functions in $\{0, 1\}$ *by associating to a bar* $\mathcal{B}$ *the function* π *such that* $\pi(q) = 1$ *if and only if there exists* $r \in \mathcal{B}$ *such that* $q \geq r$.

A natural question arises: if one code as usual conjunction, disjunction and existantial using implication and second order universal quantification what semantics is induced by this coding ? If we keep the original conjunction, disjunction and existantial, it is obvious that the defined connective are provably equivalent to the original one, and therefore, have the same semantics. However, if we remove conjunction, disjunction and existantial from the model we only have the following:

Proposition 5.2 *The semantics induced by the second order coding of conjunction, disjunction and existantial is the standard Kripke's semantics if the model is full (that is if* $\Pi_{n,p}$ *is the set of all increasing functions with the desired properties).*

proof:

$A \wedge B = \forall X((A \to (B \to X)) \to X)$: We must prove that $\mathcal{K}_2, \sigma, p \Vdash A \wedge B$ if and only if $\mathcal{K}_2, \sigma, p \Vdash A$ and $\mathcal{K}_2, \sigma, p \Vdash B$. The right to left implication is trivial. For the left to right, we assume $\mathcal{K}_2, \sigma, p \Vdash A \wedge B$, We consider the interpretation π defined by $\pi(q) = 1$ if and only if $\mathcal{K}_2, \sigma, q \Vdash A$ and $\mathcal{K}_2, \sigma, q \Vdash B$. Then it is immediate that $\mathcal{K}_2, \sigma[X := \pi], p \Vdash A \to (B \to X)$. So we have $\mathcal{K}_2, \sigma[X := \pi], p \Vdash X$ which means that $\pi(p) = 1$ which is equivalent to $\mathcal{K}_2, \sigma, p \Vdash A$ and $\mathcal{K}_2, \sigma, p \Vdash B$.

$A \vee B = \forall X((A \to X) \to (B \to X) \to X)$: The proof is similar using π defined by $\pi(q) = 1$ if and only if $\mathcal{K}_2, \sigma, q \Vdash A$ or $\mathcal{K}_2, \sigma, q \Vdash B$.

$\exists \chi\, A = \forall X(\forall \chi(A \to X) \to X)$: The proof is similar using π defined by $\pi(q) = 1$ if and only if there exists ϕ a possible interpretation for χ such that $\mathcal{K}_2, \sigma[\chi := \phi], q \Vdash A$. □

If we compare this proof to the proof in [3, 4] about data-types in AF2, we remark that second order intuitionnistic models are very similar to realizability models. Moreover, in both cases, we are in general unable to compute the semantics of a formula if the model is not full (for realizability, not full means that the interpretation of second order quantification is an intersection over a strict subset of the set of all set of lambda-terms).

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