ON A QUESTION OF SUPPORTS

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Abstract. We give a sufficient condition in order that n closed connected subsets in the n-dimensional real projective space admit a common multitangent hyperplane.

1. Introduction

The motivation for the present note is a step in the proof of the following statements [JPM04, Corollary 5.5 and Theorem 6.1] or [Man17, Man20, ğ5.3]:

Theorem 1. *Let X be a real del Pezzo surface of degree 2 such that $X(\mathbb{R})$ is homeomorphic to the disjoint union of 4 spheres. Then a smooth map $f\colon X(\mathbb{R}) \to \mathbb{S}^2$ can be approximated by regular maps if and only if its topological degree is even.*

Theorem 2. *Let X be a real del Pezzo surface of degree 1 such that $X(\mathbb{R})$ is homeomorphic to the disjoint union of 4 spheres and a projective plane. Then every smooth map $f\colon X(\mathbb{R}) \to \mathbb{S}^2$ can be approximated by regular maps.*

In the statements above $\mathbb{S}^2 \subset \mathbb{R}^3$ is the real locus of the quadric $x_1^2+x_2^2+x_3^2=1$ and a *regular map* is only regular on real algebraic loci, see [Man17, Man20, Definitions 1.2.54 and 1.3.4] for details.

One key point in the proof of the former statements was the existence of a bitangent line to any pair of connected components of a plane quartic and the existence of a tritangent conic to any triple of connected components of certain space sextic. To be precise we need the following:

Proposition 3. *Let $n = 2, 3$ and $X \subset \mathbb{P}^n$ be a smooth real algebraic curve of degree $2n$ whose real locus $X(\mathbb{R})$ has at least $n + 1$ connected components. If $n = 3$, assume furthermore that X lies on a singular quadric.*

Choose n connected components $\Omega_1, \dots, \Omega_n$ of $X(\mathbb{R})$. Then there exists a hyperplane of $\mathbb{P}^n(\mathbb{R})$ which is tangent to Ω_i for all $1 \leqslant i \leqslant n$.

Given a pair of embedded circles in the plane, it seems rather clear that a line tangent to each of them exists provided that the circles are unnested. Anyway, finding a rigorous proof of this is not straightforward and we did not find proper reference in the literature. It's less obvious to find a tritangent conic to three embedded circles in a cone. More generally, we can wonder how to generalize the obvious necessary condition to be unnested in a more general setting and, even better we can seek for a necessary and sufficient condition. We find a sufficient (but still not necessary) condition in a rather general setting. This is the main result of this short note (Theorem 10) from which we derive easily Proposition 3 as a particular case. Sections 2 and 3

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are devoted to the proof of this theorem. In Section 3, we prove Proposition 3 and propose a conjecture with a sufficient condition weaker than Theorem 10. We refer to the cited references for the proofs of Theorems 1 and 2.

2. Some reminders

We start with some well-known definitions from convex geometry.

Definition 4 (Convex hull)**.** Let E be an Euclidean space of dimension n. A subset $A \subset E$ is called *convex* in E if and only if for all $x, y \in A$ and every $t \in [0,1]$ we have

$$tx + (1-t)y \in A,$$

i.e. the line segment joining x and y is contained in A. The *convex hull* of a subset $A \subset E$ is the smallest (in the inclusion sense) convex subset of E containing A.

Definition 5 (Extremal point)**.** Let E be an Euclidean space of dimension n and $A \subset E$ be a subset. We say that a point $x \in A$ is an *extremal point* of A if the convex hull of $A \setminus \{x\}$ is still convex.

Theorem 6 (**Krein-Milman**)**.** *Every non-empty compact convex subset of a Euclidean space admits an extremal point.*

Proof. See for instance [Bou53, Chap. II.4 Th. 1]. ☐

Corollary 7. *Every non-empty compact subset of a euclidean space admits an extremal point.*

Proof. Let A be a non-empty compact subset of a Euclidean space. Let A_c be the convex hull of A. By Krein-Milman, there exists an extremal point $x \in A_c$. If $x \notin A$, then the convex set $A_c \setminus \{x\}$ contains A and it is a strict subset of A_c, which contradicts A_c being the convex hull of A. Therefore, $x \in A$. ☐

3. n-SUPPORTING HYPERPLANES

Definition 8 (Supporting hyperplane)**.** Let H be a hyperplane of a Euclidean space E given by the equation $l(x) = a$, where l is a linear form and $a \in \mathbb{R}$. We denote by H^+ and H^- the half-spaces

$$H^+ := \{x \in E \mid l(x) \geq a\} \qquad H^- := \{x \in E \mid l(x) \leq a\}.$$

Let $A \subset E$ be a subset of E and $x \in A$. We say that H is a *supporting hyperplane* of A in x (or that H *leans on* A *in* x) if and only if the following hold:

1. $x \in A \cap H$
2. $A \subset H^+$ or $A \subset H^-$.

If A is a subset of $\mathbb{P}^n(\mathbb{R})$ and $x \in A$, we say that H leans on A in x if and only if there exists an affine chart E of $\mathbb{P}^n(\mathbb{R})$ such that $x \in E$ and H leans on A in x inside E.

Definition 9 (r-supporting hyperplane)**.** Let $A_1, \ldots, A_r$ be subsets of $\mathbb{P}^n(\mathbb{R})$. We say that H is a *hyperplane of* r*-support* of $A_1, \ldots, A_r$ if there exist points $x_1 \in A_1, x_2 \in A_2, \ldots, x_r \in A_r$ such that H is a supporting hyperplane of A_i in x_i for all $1 \leq i \leq r$.

Theorem 10. *Let $n \in \mathbb{N}$ and let $A_1, \ldots, A_n \subset \mathbb{P}^n(\mathbb{R})$ be closed connected subsets of $\mathbb{P}^n(\mathbb{R})$. Suppose that there exists a point $p \in \mathbb{P}^n(\mathbb{R})$ such that no hyperplane passing through p meets all the A_i. Then there exists an n-supporting hyperplane of $A_1, \ldots, A_n$.*

Proof. We write $\mathbb{P} = \mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^* = (\mathbb{P}^n(\mathbb{R}))^*$ for the dual projective space. To each hyperplane $H \subset \mathbb{P}$ given by an equation $\sum \lambda_k x_k = 0$, we associate the point $H^* := (\lambda_0 : \lambda_1 : \cdots : \lambda_n)$ in $\mathbb{P}^*$. To each point $q \in \mathbb{P}$ we associate the dual hyperplane $q^* := \{H^* \mid q \in H\}$ in $\mathbb{P}^*$.

The hypothesis that there exists a point $p \in \mathbb{P}$ such that no hyperplane passing through p meets all the A_i implies that the A_i are pairwise disjoint. Let $\mathcal{H}$ be the set of hyperplanes in $\mathbb{P}$ that meet all the A_i. Since there is a hyperplane through n points in $\mathbb{P}$, we see that $\mathcal{H}$ is non-empty. Let $\mathcal{H}^*$ be the image of $\mathcal{H}$ in the dual space $\mathbb{P}^*$ via the above correspondance. Since p^* corresponds to the set of hyperplanes in $\mathbb{P}$ passing through p, the set $\mathcal{H}^*$ is contained in the complement of the hyperplane p^* in $\mathbb{P}^*$. Let U_p be the open affine complement of p^* in $\mathbb{P}^*$.

Lemma 11. *The set $\mathcal{H}^*$ is compact in U_p.*

Proof. For each $1 \leq i \leq n$, let $\mathcal{H}_i$ be the set of hyperplanes that meet A_i. We have $\mathcal{H}^* = \cap_{i=1}^n (\mathcal{H}_i)^*$. The set A_i being closed implies that $(\mathcal{H}_i)^*$ is closed. We start by showing that the complement of $\mathcal{H}^*$ in U_p is open.

Indeed, the natural map $\mathbb{R}^{n+1} \to \mathbb{P}, (x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \cdots : x_n]$ induces a continuous double cover $\mathbb{S}^n \to \mathbb{P}$. The inverse image B_i of A_i through this map is a closed subset in the unit sphere of $\mathbb{R}^{n+1}$. If H is an hyperplane in $\mathbb{P}$ that does not meet A_i, then its preimage H' is an hyperplane in $\mathbb{R}^{n+1}$ which does not meet B_i. The intersection $H' \cap \mathbb{S}^n$ is the unit sphere of dimension $n-1$ in H' and in particular is closed in $\mathbb{S}^n$.

If $d > 0$ is the distance between the two compacts B_i and H', we can take U_i the subset of $\mathbb{P}^*$ formed by the duals of hyperplanes whose traces on $\mathbb{S}^n$ are at distance less than $\frac{1}{2}$ of B_i. Then $U_i \setminus \{p\}$ is open in U_p.

This shows that the complement of $(\mathcal{H}_i)^*$ in $\mathbb{P}^*$ is open. It follows that $\mathcal{H}^*$ is closed in $\mathbb{P}^*$. Moreover, the set $\mathcal{H}^*$ is bounded in U_p because it is closed and $\mathcal{H}^* \cap p^* = \varnothing$. Hence $\mathcal{H}^*$ is compact in U_p. ☐

By Corollary 7 of Krein-Milman and Lemma 11, the set $\mathcal{H}^*$ admits an extremal point H^*. Let us show that H is an n-supporting hyperplane of $A_1, \ldots, A_n$.

We proceed by contradiction and without loss of generality, we can suppose that H does not support A_1. Since $H \in \mathcal{H}$, there exists for each $i = 2, \ldots, n$ a point $y_i \in A_i \cap H$. Let P_1 be a hyperplane passing through p and $y_2, \ldots, y_n$ and recall that P_1 does not meet A_1 by hypothesis. Since H does not lean on A_1, it does not lean on A_1 in the affine chart $E = \mathbb{P} \setminus P_1$. We place ourselves inside E. The hyperplane $H \cap E$ defines two half-spaces H^+ and H^- in E and there exists $x_1 \in A_1 \cap H^+ \setminus H$ and $x_2 \in A_1 \cap H^- \setminus H$.

Let S be the closed segment $[x_1, x_2]$ in E. It intersects H. Let us show that

(1) any hyperplane in E that meets S also meets A_1.

Let P be a hyperplane of E meeting S. If it meets S in x_1 or x_2, we are finished. Suppose that $P \cap S \subset]x_1, x_2[$ and $A_1 \cap P = \varnothing$. Let $O^+ = P^+ \setminus P$ and $O^- = P^- \setminus P$. The sets O^+ and O^- are open subsets of E and $A_1 \subset O^+ \cup O^-$. The subspace A_1 being connected in E, we have $A_1 \subset O^+$ or $A_1 \subset O^-$. This is impossible because $x_1 \in O^+$ and $x_2 \in O^-$ (or the other way around). this ends the proof of (1).

Let $y \in S$. Since $y_2, \ldots, y_n$ are pairwise distinct and are not contained in E (remember that $y_i \in A_i \cap P_1$ for $i \in \{2, \ldots, n\}$ by definition of P_1) and $S \subset E$, there exists a hyperplane $H_y \subset \mathbb{P}$ through $y, y_2, \ldots, y_n$. The hyperplane H_y is contained in $\mathcal{H}$ because it meets A_1 by property (1).

The points $y_2, \dots, y_n$ define a line D in $\mathbb{P}^*$ and we have $(H_y)^* \in D$. Therefore, the set of $(H_y)^*$, $y \in S$, is a closed segment S^*. It is contained in U_p, because $p \notin H_y$, and S^* is contained in $\mathcal{H}^*$ as a consequence of (1). Let $y_0 = S \cap H$, where H^* is the extremal point of $\mathcal{H}^*$ from above. Then $H^* = (H_{y_0})^*$ is a point in the interior of S^*. It is therefore contained in the convex hull of $\mathcal{H}^*$ and cannot be an extremal point, because we lose convexity if we take it away. Hence the contradiction. ☐

4. Conclusion

Proof of Proposition 3. First recall that any hyperplane meets any connected component of $X(\mathbb{R})$ in an even number of intersection points, counted with multiplicity, see e.g. [Man17, Man20, Lemma 2.7.8]. Let p be a point of $X(\mathbb{R}) \setminus \cup \Omega_i$. By definition of the degree, a hyperplane passing through p cannot meet n other components of $X(\mathbb{R})$ because X has degree $2n$ in $\mathbb{P}^n$.

The conclusion follows from Theorem 10. ☐

Theorem 10 is enough to prove Proposition 3, but it's easy to see that the existence of a point p such that no hyperplane passing through p meets all the A_i is not necessary. Take for example two intersecting circles in the plane.

We propose the following conjecture using a weaker sufficient condition (which can be applied to the former example):

Conjecture 12. *Let $\{A_i\}_{1 \le i \le n}$ be closed connected subsets contained in an affine subset of $\mathbb{P}^n(\mathbb{R})$. Let C_i be the union of all $(n-2)$-dimensional linear subspaces $P \subset \mathbb{P}^n(\mathbb{R})$ such that for all $j \neq i$, $1 \le j \le n$, P meets the convex hull of A_j. Assume that for all $1 \le i \le n$, A_i is not included in interior of C_i, then there exists an n-supporting hyperplane of $A_1, \dots, A_n$.*

Remark that this new sufficient condition is still unnecessary: consider three disjoint spheres A_1, A_2 and A_3 with the same radius and whose center are on the same line. If A_1 is not the sphere in the middle it is in the interior of the union of all lines meeting A_2 and A_3.

We can see that the sufficient condition of the conjecture is weaker than the one of Theorem 10, by contraposition. If the condition of the conjecture is not satisfied, then there exists i such that A_i is included in the interior of the union of the $(n-2)$-dimensional linear subspaces meeting each convex hull of A_j, $j \neq i$. Then there exists a $(n-2)$-dimensional linear subspace P meeting all A_i. Let $p \in \mathbb{P}$, then the hyperplane generated by p and P meet all A_i which contradicts the condition of the theorem.

We could also ask about the number of multi-tangent planes.

Proposition 13. *Under the conditions of Theorem 10, if each A_i contains a non empty open subset, then there is at least $n+1$ distinct a n-supporting hyperplanes of $A_1, \dots, A_n$.*

Proof. If each A_i contains a non empty open subset, so does $\mathcal{H}^*$. This implies that there is at least $n+1$ distinct extremal points for $\mathcal{H}^*$. Indeed, if $\mathcal{H}^*$ as less than $n+1$ extremal points, it is the convex-hull of its extremal points and therefore it is an hyperplane of dimension at most $n-1$ hence does not contain any open set. Then, the proof of theorem 10 establishes that each extremal points for $\mathcal{H}^*$ corresponds to a distinct n-supporting hyperplanes. ☐

However, it seems that the conditions of this theorem implies that we have 2^n extremal points (in dimension 2: 4 bitangent lines, 8 in dimension 3, etc.) By going either below or above each A_i. This suggest that $\mathcal{H}^*$ ressemble to a cube. Moreover, all the examples we studied lead us to propose the following conjecture.

Conjecture 14. *The main condition of Theorem 10 is sufficient and necessary to have* 2^n *multi-tangent planes when the* A_i *are not thin (i.e. contain an open subset).*

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