Isotopic piecewise affine approximation of algebraic or C^1 varieties

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Abstract

We propose a novel sufficient condition establishing that a piecewise affine variety has the same topology as a variety of the sphere $\mathbb{S}^n$ defined by positively homogeneous C^1 functions. This covers the case of C^1 varieties in the projective space $\mathbb{P}^n$. We prove that this condition is sufficient in the case of codimension one and arbitrary dimension. We describe an implementation working for homogeneous polynomials in arbitrary dimension and codimension and give experimental evidences that our condition might still be sufficient in codimension greater than one.

1 Introduction

1.1 Contribution

Let a variety V be defined by a system of implicit equations: $V = \{x \in \mathbb{K}, f_1(x) = 0, \dots, f_m(x) = 0\}$ on some compact polyhedron $\mathbb{K} \subset \mathbb{R}^n$, with $1 \leq m \leq n$. We assume that the functions $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are C^1. Let $S = (S_i)_{i \in I}$ be a decomposition of $\mathbb{K}$ into a family of simplices. A piecewise affine variety may always be defined from V and S by defining for each $1 \leq i \leq k$ an approximation $\tilde{f}_i$ of the function f_i by

- $\tilde{f}_i(x) = f_i(x)$ for any x vertex of S_i for $i \in I$ and
- $\tilde{f}_i$ is affine when restricted to any S_i for $i \in I$.

From this, we define $\tilde{V} = \{x \in \mathbb{K}, \tilde{f}_1(x) = 0, \dots, \tilde{f}_k(x) = 0\}$. The question is to find a sufficient condition ensuring that V and $\tilde{V}$ are isotopic.

Moreover, we search for a criteria that can be computably approximated with arbitrary precision in the case of multivariate polynomials, to allow for an implementation.

We propose two theorems in codimension one ($m = 1$) and two conjectures, one weaker than the other, supported by some experimental evidence, in the general case. More precisely:

- In section 3, we give a theorem that answers the question when $\mathbb{K}$ is a compact polyhedron in $\mathbb{R}^n$, in codimension one ($m = 1$) and when f_1 is of C^1 class.
- In section 4, we show that the same condition is correct if $\mathbb{K} = \mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$, in codimension one and when f_1 is of C^1 class and positively homogeneous of degree d (i.e. $f(\lambda x) = \lambda^d f(x)$ for all $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^{n+1}$). This case could be considered as codimension 2, but the homogeneity allows to ignore completely the equation of the sphere.
- In section 5, we generalise the previous statement to arbitrary codimension and conjecture that it still holds. This conjecture is supported by an implementation which was tested on many examples and never gave wrong topology. We actually give two conjectures, because we lack some results on convex set of full rank matrices. We give more details in subsection 1.5 and section 5.

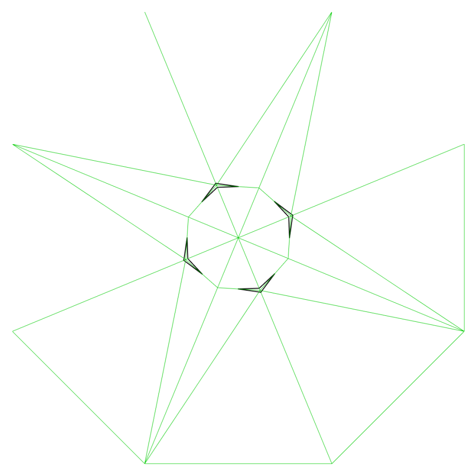

Figure 1: piecewise linear approximation of a quartic curve

Let give now the statement of our first theorem in section 3:

Theorem 2 page 8. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a compact polyhedron. Let $(S_i)_{i\in I}$ be a simplicial decomposition of $\mathbb{K}$. Let $f : \mathbb{R}^n \to \mathbb{K}$ be a C^1 function in n variables. Let $V = \{x \in \mathbb{K}, f(x) = 0\}$ be the zero locus of f restricted to $\mathbb{K}$. Assume that $V \cap \partial\mathbb{K} = \emptyset$.*

We define $\tilde{p} : \mathbb{K} \to \mathbb{R}$ the piecewise affine function such that for all $i \in I$, $\tilde{f}|_{S_i}$ is affine and for any v vertex of S_i, we have $f(v) = \tilde{f}|_{S_i}(v)$. We define the following:

- $\tilde{V} = \{x \in \mathbb{K}, \tilde{f}(x) = 0\}$ *the zero locus of* $\tilde{f}$.
- $\mathbb{K}(f) = \{x \in \mathbb{K}, f(x)\tilde{f}(x) \leq 0\}$.
- $\tilde{\nabla} f(x) = \{\nabla \tilde{f}|_{S_i}(x), x \in S_i\} \subset \mathbb{R}^n$
- $G(f, x) = \{\nabla f(x)\} \cup \tilde{\nabla} f(x) \subset \mathbb{R}^n$

If the condition (1) below holds, then V and $\tilde{V}$ are isotopic:

$$\forall x \in \mathbb{K}(f), 0 \notin \mathcal{H}(G(f, x)) \text{ the convex hull of } G(f, x) \tag{1}$$

Let us give some ideas about this theorem: the isotopy is naturally defined by $f_t(x) = tf(x) + (1-t)\tilde{f}(x)$ for $x \in \mathbb{K}$ and $t \in [0, 1]$. The function $x \mapsto f_t(x)$ is not differentiable, but it is differentiable in any direction, and its gradient at x in direction D is always given by a scalar product $V.D$ where V is in the convex hull of exactly the gradients we are considering in the set $G(f, x)$. Thus our condition ensures that $V \neq 0$, which we find is very natural *smoothness* condition. This condition only needs to hold in region where f_t may be null, i.e. when $f(x)$ and $\tilde{f}(x)$ have opposite sign, this justifies the definition of $\mathbb{K}(f)$.

The theorem of section 4 is almost the same, we ask for the function to be positively homegenous and we decompose $\mathbb{R}^{n+1}$ in *simplicial cones*, which is defined in section 2. Appart from this, the statement is unchanged.

In section 5 we propose two conditions (conjecture 4 and 8) that could apply in arbitrary codimension (i.e. with more than one polynomials). Unfortunately we are not able to prove those conjectures.

The first one (conjecture 4), the most natural, generalises the condition (1)

$$\forall x \in \mathbb{K}(p), 0 \notin \mathcal{H}(G(p, x))$$

into

$$\forall x \in \mathbb{K}(p), \forall A \in \mathcal{H}(G(p,x)), A \text{ is of maximal rank} \tag{2}$$

This is natural as with codimension greater than one, the gradients become matrices and maximal rank expresses transversality, hence smoothness.

Remarks: we do not need extra hypotheses, like smoothness (or non complete intersection with codimension greater than one). However, if the variety is not sm ooth, our criteria will never be satisfied. We should also say that our condition is frame independant. Indeed, a change of coordinates will multiply all the gradients by the same invertible matrix and the convex hull is transformed accordingly.

1.2 A global criteria

A standard way to compute piecewise affine approximation of varieties defined by implicit equations are decomposition method that proceed by incrementally subdivising the ambient space in simplices or hypercubes, until some criteria is met.

Our criteria is such a stopping condition, but it is global. To our knowledge, all existing criteria (like in [16]) will ensure the isotopy of the orginal variety and its approximation when restricted to each simplex or hypercube. This is not the case of our criteria.

Let us explain more precisely what we mean by *global*. From the definitions in our theorem or conjecture, if follows that if X is the set of vertices of the simplicial decomposition, we have $X \times V$ isotopic to $X \times \tilde{V}$. This means that the isotopy can not traverse the vertices of the decomposition. However it may traverse faces of simplices of dimension one or more, allowing to use less simplices.

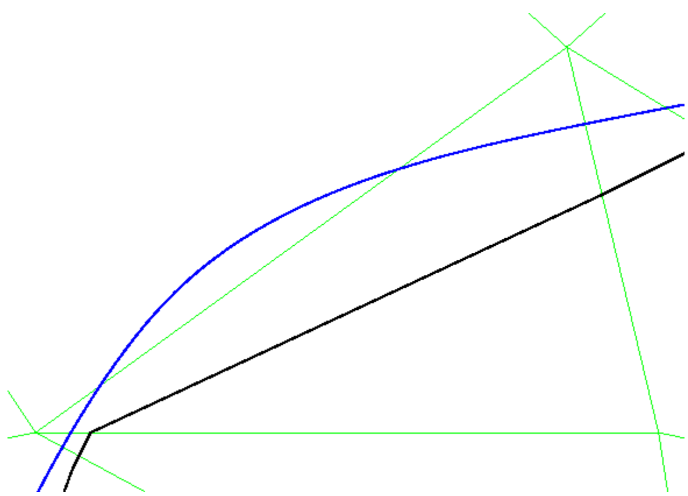

Figure 2: Our criteria is global

This is illustrated by figure 2. This figure represents a piece of a C^1 variety in blue and its approximation in black. In the triangle which is fully displayed, the approximation has only one component while the original variety, has two components. Still our criteria accepts this decomposition. This can save quite a lot of triangles.

1.3 Testing the criteria

Implementing a test for the criteria is not possible in general. But, in the case of polynomials, we can use Bernstein basis: for all x in a simplex S, $\nabla p(x)$ always lies in the convex hull of the coefficients of the polynomial ∇p, in this basis, after a change of variable to send the unit

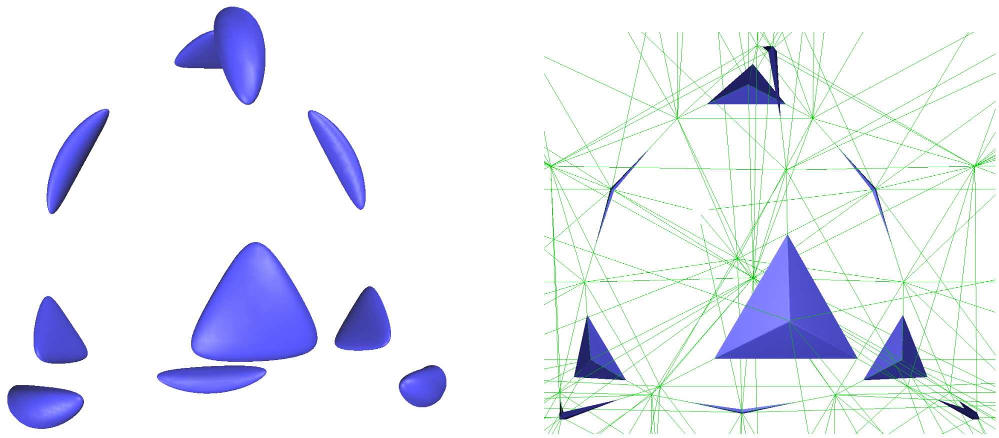

Figure 3: a quartic surface and its piecewise affine approximation

simplex in S. This gives easily a sufficient condition to satisfy the test in each face of the simplicial decomposition. This is detailed in section 6.

Moreover, we can approximate the real criteria with arbitrary precision by subdividing each face to test. We only do this subdivision for the test. Refining the decomposition requires to subdivise the neighbour simplices and we do not want to do that if we can avoid it.

We implemented a heuristic that searches for a simplicial decomposition satisfying our criteria. This is relatively quick because it uses floating point arithmetic. But when an apparently correct decomposition is found, we retest the criteria with exact rational arithmetic, ensuring the correctness.

Moreover, the search for the decomposition produces certificates that the relevant convex hulls do not contain 0. In codimension one, such a certificate is a vector which has a positive scalar product with all the generators of each convex hull. This way the only computation we have to do in exact arithmetic are change of coordinates, scalar products and comparison. We do not perform nested computation in loops and this limits the growth of the size of numerators and denominators. In our experiments, the computing time of the final exact test is faster than the search for a simplicial decomposition.

1.4 Examples

Remark : Because polynomials have no well defined value in the projective space, we will work within $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. This is equivalent and much easier. Still all the examples will be depicted in the projective space as it avoids to draw every point of the variety twice.

The green line segments in figures 1 and 3 are the edges of a simplicial decomposition of $\mathcal{P}^2(\mathbb{R})$ and $\mathcal{P}^3(\mathbb{R})$ respectively (in the latter case, it is unfortunately not easy to guess the simplices from their edges). The figure 1 gives the piecewise affine approximation of a plane curve of degree 4. It uses a decomposition of the projective plane with 13 vertices and 24 triangles and requires 58ms to compute. The figure 3 shows an algebraic surface of the same degree together with its approximation. The decomposition uses 32 vertices and 152 tetrahedron and requires 3.3s to compute.

As those varieties are enclosed in a compact polyhedron, their approximations can be proved isotopic to the original varieties by the previous theorem 2. They can also be proved correct by the theorem 3 of section 4.

1.5 Another conjecture

A key ingredient for both the proof and the implementation is the geometric form of Hahn-Banach theorem: in codimension 1, if S is a finite set of vectors, from $0 \notin \mathcal{H}(S)$, we get a vector N such that $N.V > 0$ for all $V \in S$. Unfortunately, we could not find nor prove a similar result for convex sets of full rank matrices. This suggests the following very interesting conjecture:

Conjecture 5. *Let $1 < m \leq n$ be two natural numbers, let $S \subset \mathrm{Mat}_{m,n}(\mathbb{R})$ be a convex set of matrices of rank m. There exists a matrix $M \in \mathrm{Mat}_{m,n}(\mathbb{R})$ such that $M^tA + A^tM$ is symmetric definite and positive for all $A \in S$.*

This Hahn-Banch conjecture also allows for a notion of *certificate* allowing to search for a decomposition using efficient floating point computations, and rechecking the final result using exact rational arithmetic. We can check quickly that $M^tA + A^tM$ is symmetric definite and positive using Choleski decompostion for each matrices A in S (if S is finite). This does not require to compute the spectrum of the matrix.

Unfortunately, we do npt know how to implement a test to decide if all matrices in a convex set of matrices are fullrank (used in 2), that produces a certificate. A constructive proof of conjecture 5 would likely provide such a test.

To be able to propose an implementation working in codimension greater than one, we use a simpler sufficient condition, using only scalar products, that implies (2). This means we have stronger evidence for another conjecture 8 using this stronger condition than for conjecture 4 using (2).

1.6 Search for a simplicial decomposition

Our implementation is described in section 6 and is available on github:

`https://github.com/craff/hypersurfaces`.

It implements a semi-algorithm building a simplicial decomposition satisfying our criteria. This means we can solve non-degenerate systems of homogeneous real polynomial equations. Non-degenerate means that the jacobian matrix of the system is full rank at point that are in the solution. By *solving* we mean finding a piecewise affine approximation of the solution that is isotopic to the real solution. It allows to compute topological invariants of the solution and in particular the number of connected components and the Betti numbers of each component (see for instance the appendix B.3 of [17] for definitions). It is a semi-algorithm, which means that it may loop when the system of polynomials is degenerated. Our semi-algorithm terminates, in principle, if we know that the system of polynomials is non-degenerated.

In codimension one, our implementation is exact and provides a proved result about the hypersurface that reposes only on the correction of the final test using the certificate. This test is a rather short piece of code and would be easy to rewrite. For codimension greater than one, the validity of the answer of our algorithm depends upon conjecture 8.

The search for an adequate simplicial decomposition is in an early stage. Still some examples are already quite interesting. For instance, we are able to compute Betti numbers of random cubic given by a random homogeneous polynomials up to dimension 6 in less than one hour. Moreover, to our knowledge, there is no available exact implementation working in arbitrary dimension and codimension.

1.7 Related work

1.7.1 Viro's method

Our main result may be seen as some inverse of Viro's method [22, 18], more precisely the combinatorial patchworking version for complete intersection [20]. Indeed, Viro's method allows for the construction of polynomials with a zero locus having the same topology than a given piecewise linear variety. We do the opposite.

An important difference is that we use arbitrary simplicial decomposition rather than the Newton polytope of polynomials of a chosen degree, reflected in each orthant. Due the fact that we do not use only polynomial and therefore Newton Polytope, we did not manage to use [11, 19] to shorten the proof.

1.7.2 Decomposition methods using Descartes' rule of sign

For univariate polynomials, there is a similar available criteria: Descartes' rule of sign. It allows to ensure that a polynomial as at most one root in a given interval. There are many attempts to generalise this rule to the multivariate case. Most of these work [10, 14, 4] only consider the zero dimensional case. They give an upper bound of the number of solutions. If this upper bound is one, it would allow to isolate each solution. A counter-example to Roy and Itenberg conjecture [10] given by Li and Wang [15], means that even for the zero dimensional case, there is no known generalisation of Descartes' rule of sign that works in all cases. In a recent work [8] Descates rules of sign is generalised to arbitrary hypersurfaces, but only give a bound to the number of components which is not enough to compute the topology of hypersufaces.

1.7.3 Other decomposition methods

Our approach is similar to many other algorithms that work by subdivising the ambient space in hypercubes or simplices. See [1] for a book covering most algorithms in the domain. In dimension 3, Marching cube algorithms usually ensure correct topology between the piecewise trilinear function given by the value at the vertex of each hypercube and the produced piecewise affine variety [21, 9]. There are not many algorithms for hypersurfaces in arbitrary dimension, we may cite [6].

For works that provide exact algorithms using decomposition, in the case of polynomials for 2D or 3D curves or 3D surfaces, we may cite [7, 2]. Some of these works, like [16] also use Bernstein basis in the implementation. This latter also uses Descartes' rule of sign of the partial derivative of the polynomials.

As we mentioned it previously, we think a key property of our criteria to stop a decomposition is its global nature. Moreover, we are not aware of any criteria that would work in arbitrary dimension and codimension.

1.7.4 Algebraic methods

Other algorithms, which are exact to compute topological invariants of algebraic variety, are roughly based on the decidability of the theory of real closed fields. Thus, they are more *algebraic*, using cylindrical decompositions, Groebner basis, resultants, ... They have the main advantage of being able to deal with arbitrary singularities, but are of a very different nature. Moreover, very few of these algorithms have free implementations and we could not compare them with our algorithm, for instance on the computation of Betti numbers of random cubic in dimension up to 6.

Some available implementations, that could fit in this *algebraic* category, are limited to curves and surfaces like `bertini_real` [5] or to the zero dimensional case like `msolve` [3]. Those two are probably the best ones available. It is worth noticing that, unlike most available implementation for curve and surfaces, `bertini_real` does not limit the number of variables and allow to compute a surface embedded in a space of high dimension. However, unlike `msolve`, `bertini_real` is only using arbitrary precision floating point arithmetic and is not an exact algorithm.

Currently the above cited algorithms support some management of singularities and seem faster than our algorithm. This is to be expected as it is natural that more general algorithms are slower than specialised ones. Moreover, our search for a simplicial decomposition that satisfies our criteria is in an early stage. We outline below some directions of research that could allow our criteria to be used in an algorithm that could compete with the state of the art algorithms.

Remark: a lot of the algorithms found in the literature are not freely available or hard to install. We only succeeded to install `bertini_real` and `msolve`!

1.8 Further theoretical research

Apart from proving or disproving the afford mentioned conjectures, it would be nice to provide a complexity analysis for our algorithm. This will require (as for some algorithms searching for roots of univariate polynomials), a measure of regularity of the system of polynomials. This bound will probably be very bad because it will assume the worst everywhere, but it would still be interesting.

We would like to extend our work to product of projective spaces, weighted projective spaces or compact of $\mathbb{R}^n$ with a border condition allowing the variety to meet the border.

For singular varieties, it is likely that our criteria is still correct for singularities which are affine subvariaties, provided that all singularities of dimension m are entirely covered by some faces of the simplicial decomposition of dimension m. For instance, isolated points must be among the vertices of the decomposition. Then, the only modification of our criteria is to ignore the singular faces and it seems to work. We tested this on isolated singularities, and this seems to work. Singularities which are not affine subvarieties seem must more challenging.

1.9 Further implementation research

The main problem with the current implementation is that we do not know yet what are the best triangulations for our criteria, especially in the case of codimension greater than one. For instance, if we impose the vertices of the decomposition, what is the best triangulation to try to meet the criteria? Currently we use the convex hull of the vertices projected on the unit sphere. This is similar to Delauney's triangulation. This is frame independant, but there is no reason that such triangulations are the best to meet our criteria for a given polynomial system. The same is true for the choice of vertices. Currently, we favour critical points as it seems to give good results, but this is not frame independant and does not always give enough points and we don't really known what other points to choose.

We should also note that our implementation is written in OCaml using functors to parameterise the implementation by the representation of numbers, because it allows for rapid prototyping. A C implementation optimised for speed could gain a factor 2 or 3 and parallelisation could allows to gain a factor 10 and should be possible using OCaml 5.

We think it is possible to reach computing time matching those of existing decomposition algorithms for curves and surfaces.

Another way to improve the efficiency is to combine our criteria with variables elimination techniques. An idea would be to perform *easy* eliminations, before using our algorithm. For instance, one could eliminate a variable if it occurs only in one monomial of some polynomials. The current implementation is not even doing elimination of linear equations! But this is planed.

1.10 Thanks

We thank Stéphane Simon for showing us his marching cube implementation, 25 years ago, starting our interest in this research topic. We thanks Ilia Itenberg for several discussions, in particular about Viro's method. Finally, we heartily thanks Frédéric Mangolte for the lengthy discussions on this research, his comments and great help.

2 Notation and convention

Here are a few notation we use:

- $\mathcal{H}(S)$ denotes the convex hull of a subset S of $\mathbb{R}^n$.
- When $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in all direction, we denote $\nabla(f)(x)(v)$ the differential of f at x in the direction v. In general, we only have $\nabla(f)(x)(\lambda v) = \lambda\nabla(f)(x)(v)$ for $\lambda > 0$ as $v \mapsto \nabla(f)(x)(v)$ may be non linear.
- When $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable $\nabla(f)(x)$ will denote the $m \times n$ Jacobian matrix.

Simplicial decomposition In section 3, we consider simplicial decomposition of a compact polyhedron $\mathbb{K} \subset \mathbb{R}^n$. By simplicial decomposition, we mean, as in [18], a family of simplices $(S_i)_{i\in I}$ such that:

- $\mathbb{K} = \cup_{i\in I} S_i$
- $\forall i, j \in I, i \neq j, S_i \cap S_j$ is a simplex of dimension at most $n-1$ which is the common face of S_i and S_j.
- $\forall i, j \in I, i \neq j, \mathring{S}_i \cap \mathring{S}_j = \emptyset$.

Decomposition in simplicial cones

Definition 1. *A* simplicial cone*, is a set S, that is defined from a simplex S' that do not contain 0 by*

$$S = \{\lambda x, x \in S', \lambda > 0\}$$

In section 4 and after, we consider decomposition in simplicial cones of $\mathbb{R}^{n+1}$. We mean a family of simplicial cone $(S_i)_{i\in I}$ such that:

- $\mathbb{K} = \cup_{i\in I} S_i$
- $\forall i, j \in I, i \neq j, S_i \cap S_j$ is a simplicial cone of dimension at most n which is the common face of S_i and S_j.
- $\forall i, j \in I, i \neq j, \mathring{S}_i \cap \mathring{S}_j = \emptyset$.

Bernstein basis In section 6 we refer to Bernstein basis. In the case of homogeneous polynomials of degree d, it is

$$\left(\frac{d!}{\alpha!}x^\alpha\right)_{\alpha\in\mathbb{N}^d, \Sigma_i\alpha_i=d} \text{ where } x^\alpha = \Pi_i x_i^{\alpha_i} \text{ and } \alpha! = \Pi_i \alpha_i!$$

The key property of Bernstein basis it that its value in the unit simplex lies in the convex hull of the coefficients. It is an immediate consequence of De Casteljau algorithm to compute the value of the polynomial as a barycenter. We also use this property with the gradient of a polynomial, seen as a polynomial whose coefficients are vectors.

3 Hypersurfaces on a compact polyhedron

Here is a first theorem for an hypersurface which is enclosed in the interior of a compact polyhedron of $\mathbb{R}^n$. This hypothesis seems essential and unnatural, but will disappear when we consider the entire projective space of dimension n.

Theorem 2. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a compact polyhedron. Let $(S_i)_{i\in I}$ be a simplicial decomposition of $\mathbb{K}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ of class C^1. Let $V = \{x \in \mathbb{K}, f(x) = 0\}$ be the zero locus of f restricted to $\mathbb{K}$. Assume that $V \cap \partial\mathbb{K} = \emptyset$ (3).*

We define $\tilde{f} : \mathbb{K} \to \mathbb{R}$ the piecewise affine function such that for all $i \in I$, $\tilde{f}|_{S_i}$ is affine and for any v vertex of S_i, we have $f(v) = \tilde{f}|_{S_i}(v)$. We define the following:

- $\tilde{V} = \{x \in \mathbb{K}, \tilde{f}(x) = 0\}$ *the zero locus of $\tilde{f}$.*
- $\mathbb{K}(f) = \{x \in \mathbb{K}, f(x)\tilde{f}(x) \leq 0\}$.
- $\tilde{\nabla} f(x) = \{\nabla \tilde{f}|_{S_i}(x), x \in S_i\} \subset \mathbb{R}^n$
- $G(f, x) = \{\nabla f(x)\} \cup \tilde{\nabla} f(x) \subset \mathbb{R}^n$

If the condition below holds, then V and $\tilde{V}$ are isotopic:

$$\forall x \in \mathbb{K}(f), 0 \notin \mathcal{H}(G(f,x)) \tag{4}$$

Proof. Assume the definitions and hypotheses of the theorem. For any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we define

$$G(f,x,\varepsilon) = \bigcup_{y \in \mathbb{K}(f), \|y-x\|<\varepsilon} G(f,y)$$

Remark: we need to define $G(f,x,\varepsilon)$ for $x \in \mathbb{R}^n$ because the convolution product below will cover the border of $\mathbb{K}(f)$. Clearly, for points too far from $\mathbb{K}(f)$, we have $G(f,x,\varepsilon) = 0$.

We now prove that there exists $\varepsilon > 0$ such that

$$\forall x \in \mathbb{K}(f), 0 \notin \mathcal{H}(G(f,x,\varepsilon)) \tag{5}$$

We proceed by contradiction and choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}(f)$ such that $0 \in \mathcal{H}(G(f,x_n,\frac{1}{n}))$. As $\mathbb{K}(f)$ is compact, we can assume that x_n converges to $x_\infty \in \mathbb{K}(f)$. Let us define

$$\delta = \min_{i \in I, x_\infty \notin S_i} \mathrm{dist}(x_\infty, S_i)$$

We have for any $y \in \mathbb{K}$, $\|y - x_\infty\| < \delta$ and $y \in S_i$ implies $x_\infty \in S_i$ and therefore $\tilde{\nabla} f(y) \subset \tilde{\nabla} f(x_\infty)$. Thus, for $\delta n > 1$, we have

$$0 \in \mathcal{H}(\{\nabla f(y), y \in \mathbb{K}, \|x_n - y\| < \frac{1}{n}\} \cup \tilde{\nabla} f(x_\infty))$$

The set $\{\nabla f(y), y \in \mathbb{K}, \|x_n - y\| < \frac{1}{n}\}$ converges to the singleton $\{\nabla f(x_\infty)\}$ for the Haussdorf metric and $\mathcal{H}$ is continuous for that metric. Hence,

$$\mathcal{H}(\{\nabla f(y), y \in \mathbb{K}, \|x_n - y\| < \frac{1}{n}\} \cup \tilde{\nabla} f(x_\infty)) \longrightarrow \mathcal{H}(G(f,x_\infty)) \text{ when } n \longrightarrow +\infty$$

This implies $0 \in G(f,x_\infty)$, because it is a closed set, which contradicts (4).

By the geometric form of Hahn-Banach, we can find $N : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\forall x \in \mathbb{R}^n, \forall v \in G(f,x,\varepsilon), N(x).v > 0$$

Let us choose a function $\mu : \mathbb{R}^n \to \mathbb{R}_+$ of C^∞ class, with support in the sphere of radius ε and such that $\int_{\mathbb{R}^n} \mu(u)\mathrm{d}u = 1$. We define:

$$N'(x) = N \star \mu = \int_{\mathbb{R}^n} N(u-x)\mu(u)\mathrm{d}u$$

N' is of C^∞ class on $\mathbb{R}^n$. Let us consider $v \in G(f,x)$ for $x \in \mathbb{K}$, if $\|u\| < \varepsilon$ and therefore $\|x - (u-x)\| < \varepsilon$, we have $v \in G(f, u-x, \varepsilon)$ which implies $N(u-x).v > 0$. This establishes:

$$\forall x \in \mathbb{K}(f), \forall v \in G(f,x), N'(x).v > 0 \tag{6}$$

The next step is to consider the maximal integral curves of N'. By The Cauchy-Lindelöf-Lipshitz-Picard theorem those curves exist, are unique in $\mathbb{K}(f)$ and continuous in the initial conditions.

For $t \in [0,1]$ we define $f_t : \mathbb{K} \to \mathbb{R}$, such that $f_t(x) = tf(x) + (1-t)\tilde{f}(x)$. We remark that $\tilde{f}$ is differentiable in every direction and that the differential of $\tilde{f}$ at x in the direction D is given by $D.V$ for some $V \in \tilde{\nabla} f(x)$. Remark the differential in the direction D and $-D$ may be different.

Therefore, the differential of f_t at $x \in \mathbb{K}(f)$ in the direction $N'(x)$ is given by the expression $N'(x).(t\nabla f(x) + (1-t)V)$ for some $V \in \tilde{\nabla} f(x)$ and is therefore positive by (6).

This means that the functions $f_t(x)$ are increasing along an integral curve of N' and therefore each integral curve meet the variety $V_t = \{x \in \mathbb{R}, f_t(x) = 0\}$ for $t \in [0, 1]$ in at most one point. Remark that $V_t \subset \mathbb{K}(f)$.

To finish the proof we must show that the maximal integral curves of N' have their extremity in the border of $\mathbb{K}(f)$, which are points x with either $f(x) = 0$ or $\tilde{f}(x) = 0$, by the condition (2). This is true because we can find $K > 0$ such that $\forall x \in \mathbb{K}(f), K < N'(x)$ by compacity and regularity of N'. This means that a maximal integral curve of N' will join the border of $\mathbb{K}(f)$ on an interval $[t_1, t_2]$ for $t_2 - t_1 < \frac{M}{K}$ where M is an upper bound of both f and $\tilde{f}$ on $\mathbb{K}$.

Therefore, $(x, t) \mapsto f_t(x)$ is the wanted isotopy. $\square$

4 Hypersurfaces on the projective space

We now give a condition to establish that an hypersurface in $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$ defined by a positively homogeneous C^1 function in $n+1$ variables is istopic to a variety defined by a piecewise linear function on $\mathbb{R}^{n+1}$. We state the theorem on the unit sphere $\mathbb{S}^n$ because it is simpler to write the condition than working on the projective space.

Theorem 3. *Let $(S_i)_{i \in I}$ be a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$, the unit sphere of $\mathbb{R}^{n+1}$. Let $p : \mathbb{R}^{n+1} \to \mathbb{R}$ be a positively homogeneous C^1 function of degree d. (i.e. $p(\lambda x) = \lambda^d p(x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^{\ltimes+\nVdash}$). Let $V = \{x \in \mathbb{S}^n, p(x) = 0\}$ be the zero locus of p restricted to $\mathbb{S}^n$.*

We define $\tilde{p} : \mathbb{R}^{n+1} \to \mathbb{R}$ the piecewise linear function such that for all $i \in I$, $\tilde{p}|_{S_i}$ is linear and for any v vertex of S_i, we have $p(v) = \tilde{p}|_{S_i}(v)$. We define the following:

- *$\tilde{V} = \{x \in \mathbb{S}^n, \tilde{p}(x) = 0\}$ the zero locus of $\tilde{p}$.*
- *$\mathbb{K}(p) = \{x \in \mathbb{S}^n, p(x)\tilde{p}(x) \leq 0\}$.*
- *$\tilde{\nabla} p(x) = \{\nabla \tilde{p}|_{S_i}(x), x \in S_i\} \subset \mathbb{R}^{n+1}$*
- *$G(p, x) = \{\nabla p(x)\} \cup \tilde{\nabla} p(x) \subset \mathbb{R}^{n+1}$*

If the following condition (7) holds, then V and $\tilde{V}$ are isotopic:

$$\forall x \in \mathbb{K}(p), 0 \notin \mathcal{H}(G(p, x)) \tag{7}$$

This implies that the projective varieties associated to V and $\tilde{V}$ are isotopic if the simplicial decomposition is stable by the symmetry $x \mapsto -x$.

Proof. By taking $\mathbb{K} = \mathbb{S}^n$, we can use exactly the same definitions and reasoning as the proof of theorem 2 until the definition of N' of C^∞ class satisfying

$$\forall x \in \mathbb{K}(p), \forall v \in G(p, x), N'(x).v > 0 \tag{8}$$

We must change the definition of $p_t(x)$, using d the degree of p:

$$\begin{aligned} \overline{p}(x) &= \tilde{p}(x\|x\|^{d-1}) \\ p_t(x) &= tp(x) + (1-t)\overline{p}(x) \end{aligned}$$

We have $p_t(\lambda x) = \lambda^d p_t(x)$ for all $\lambda \in \mathbb{R}_+$. As in the previous proof, $\tilde{p}(x)$ has a differential in any direction v. Hence, the functions $\overline{p}$ and p_t are differentiable in any direction, hence they satisfy Euler relation:

$$\nabla\overline{p}(x)(x) = dp(x) \qquad\qquad \nabla p_t(x)(x) = dp(x)$$

We need more precision for the derivative of $\tilde{p}$: if $x, v \in \mathbb{R}^{n+1}$ we can define $i(x,v) \in I$ such that $x \in S_{i(x,v)}$ and $x + hv \in S_{i(x,v)}$ for all $h > 0$ small enough. We also define $S(x,v) = S_{i(x,v)}$. Using this notation, the gradient of $\tilde{p}(x)$ in the direction v is given by

$$\nabla\tilde{p}(x)(x) \quad = \quad \nabla\tilde{p}|_{S(x,v)}(x).v \tag{9}$$

But as $\tilde{p}|_{S(x,v)}$ is linear, its gradient is constant and we can simply write $\nabla\tilde{p}|_{S(x,v)}.v$. Remark: $i(x,v)$ is not uniquely defined, but the choice of index in I does not change the value of the differential in a given direction.

Using these notations, we compute:

$$\begin{aligned} \nabla\overline{p}(x)(v) &= d\nabla\tilde{p}|_{S(x,v)}.v\|x\|^{d-1} \\ \nabla p_t(x)(v) &= (t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x,v)}\|x\|^{d-1}).v \end{aligned}$$

We now prove that for $x \in \mathbb{K}(p)$, $N'(x)$ is not normal to the unit sphere at x. Let us choose $x \in \mathbb{K}(p)$, we can find $t \in [0,1]$ such that $p_t(x) = tp(x) + (1-t)\overline{p}(x) = 0$ (take $t = 0$ if $p(x) = \tilde{p}(x) = 0$ and $t = \frac{\overline{p}(x)}{\overline{p}(x)-p(x)}$ otherwise). This is well defined because in $\mathbb{K}(p)$ we can only have $p(x) = \overline{p}(x)$ if $p(x) = \tilde{p}(x) = 0$. Let us define

$$V = t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x,v)}$$

The gradient $p_t(x)$ in the direction of x, is given by $V.x = p_t(x) = 0$ by Euler relation. $V \in (t + (1-t)d)G(p,x)$ and $t + (1-t)d = d - t(d-1) > 0$ for $t \in [0,1]$. This means that $N'(x)$ can not be normal to $\mathbb{S}^n$ at x as this would imply $V.N'(x) = 0$ which is impossible by (8). This ends the proof that $N'(x)$ is not normal to the unit sphere at x.

We can now define for all $x \in \mathbb{K}(p)$, $N^T(x) = N'(x) - (N'(x).x)x$, the projection of $N'(x)$ on the hyperplane tangent to $\mathbb{S}^n$ at x. For $x \in \mathbb{K}(p)$, as the polyhedra S_i are simplicial cones we can assume $S(x, N'(x)) = S(x, N^T(x))$ that we will simply write $S(x)$. Indeed, for $h > 0$ small enough, $x + hN'(x)$ and $x + hN^T(x)$ belong to the same simplicial cone S_i because they only differ by a vector in the direction of x and S_i is a cone.

Using this notation, for a point $x \in \mathbb{K}(p)$ such that $p_t(x) = 0$, the gradient of $p_t(x)$ in direction $N'(x)$ and $N^T(x)$ verify:

$$\begin{aligned} \nabla p_t(x)(N'(x)) &= (t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x)}).N'(x) \\ &> 0 \text{ by (8)} \\ \nabla p_t(x)(N^T(x)) &= (t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x)}).N^T(x) \\ &= (t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x)}).(N'(x) - (x.N'(x))x) \\ &= (t\nabla p(x) + (1-t)d\nabla\tilde{p}|_{S(x)}).N'(x) \text{ by Euler relation} \\ &> 0 \end{aligned} \tag{10}$$

Let $\gamma : J \to \mathbb{K}(p)$ be a maximal integral curve of N^T. This means $\gamma'(u) = N^T(\gamma(u))$. There are particular cases where γ is reduced to one point x when $p(x) = \tilde{p}(x) = 0$.

In all other cases, as $\tilde{p}$ is derivable in all directions, $u \mapsto \tilde{p}(\gamma(u))$ is derivable (but not necessarily of C^1 class). Similarly $u \mapsto \overline{p}(\gamma(u))$ and $u \mapsto p_t(\gamma(u))$ for $t \in [0,1]$ and $t(u) = \frac{\overline{p}(u)}{\overline{p}(u)-p(u)}$ are derivable. Moreover, a point $x \in \mathbb{K}(p)$ with $p(x) = 0$ or $\tilde{p}(x) = 0$ can only be at the extremity of γ, otherwise, by (10), γ would leave $\mathbb{K}(p)$. This means that p and $\tilde{p}$ have constant sign on γ and may be null only on the extremity. This implies that $\overline{p}(\gamma(u)) - p(\gamma(u))$ is of contant signe along such a curve γ.

We have:

$$\begin{aligned} p_{t(u)}(\gamma(u)) &= 0 \\ (p(\gamma(u)) - \overline{p}(\gamma(u)))t'(u) + \nabla p_{t(u)}(\gamma(u))(\gamma'(u)) &= 0 \end{aligned}$$

$$
\begin{array}{rcl}
(\overline{p}(\gamma(u)) - p(\gamma(u)))t'(u) & = & \nabla p_{t(u)}(\gamma(u))(N^T(\gamma(u))) \\
(\overline{p}(\gamma(u)) - p(\gamma(u)))t'(u) & > & 0
\end{array}
\tag{11}
$$

This means that $t(u)$ is monotonous along any curve γ that is not reduced to one point. As in the previous proof, $N^T(\gamma(u))$ is never null and is minored by some constant $K > 0$, thus the extremity of maximal integral curve γ will necessarily be a point where p is null and another point where $\tilde{p}$ is null.

This means that p_t defines the isotopy we are looking for. $\square$

5 Increasing codimension

We propose the following conjecture for several positively homogeneous C^1 funciton:

Conjecture 4. *Let $(S_i)_{i \in I}$ be a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$, the unit sphere of $\mathbb{R}^{n+1}$. Let $p = (p_1, \ldots, p_m)$ be a family of $m \leq n$ positively homogeneous C^1 functionx from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{\ltimes}$ of respective degree $(d_1, \ldots, d_m)$ not necessarily equal (i.e. $p_i(\lambda x) = \lambda^{d_i} p_i(x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^{\ltimes + \nVdash}$). Let $V = \{x \in \mathbb{S}^n, p(x) = 0\}$ be the zero locus of p restricted to $\mathbb{S}^n$.*

We define $\tilde{p} : \mathbb{R}^{n+1} \to \mathbb{R}^m$ the piecewise linear function such that for all $i \in I$, $\tilde{p}|_{S_i}$ is linear and for any v vertex of S_i, we have $p(v) = \tilde{p}|_{S_i}(v)$. We define the following:

- *$\tilde{V} = \{x \in \mathbb{S}^n, \tilde{p}(x) = 0\}$ the zero locus of $\tilde{p}$.*
- *$\mathbb{K}(p) = \{x \in \mathbb{S}^n, \forall i \in \{1, \ldots, m\}, p_i(x)\tilde{p}_i(x) \leq 0\}$.*
- *$\tilde{\nabla} p(x) = \{\nabla \tilde{p}|_{S_i}(x), x \in S_i\} \subset \mathrm{Mat}_{m,n+1}(\mathbb{R})$*
- *$G(p, x) = \{\nabla p(x)\} \cup \tilde{\nabla} p(x) \subset \mathrm{Mat}_{m,n+1}(\mathbb{R})$*

If the condition below holds, then V and $\tilde{V}$ are isotopic:

$$
\forall x \in \mathbb{K}(p), \forall A \in \mathcal{H}(G(p,x)), A \text{ is of maximal rank} \tag{12}
$$

This conjecture is the natural generalisation of theorem 3 and our implementation described in the next section suggest that it might be true.

Unfortunately, to adapt the proof of the previous section, we need a result analogous to Hahn-Banach for convex set of full rank matrices. This would give the countepart of the vector $N(x)$ in the previous proof and also the certificate we need for the implementation.

Here is the expected result that seems unknown and that we could not prove neither disprove:

Conjecture 5. *Let $1 < m \leq n$ two natural numbers, let $S \subset \mathrm{Mat}_{m,n}(\mathbb{R})$ a convex set of matrices of rank m. There exists a matrix $M \in \mathrm{Mat}_{m,n}(\mathbb{R})$ such that $M^t A + A^t M$ is symmetric definite and positive for all $A \in S$.*

We could not prove that conjecture 5 implies conjecture 4 but we feel that it is a key element of the proof. A way to prove this implication would be to ensure that $N^T : \mathbb{K}(p) \to \mathrm{Mat}_{m,n+1}(\mathbb{R})$ satisfies the Schwartz condition. This means we should have that the derivative of N_i^T in the direction N_j^T should be equal to the derivative of N_j^T in the direction N_i^T. Then, we could construct unique integral hypersurfaces of N^T and probably finish the proof. To to this, an idea if to build the kernel used in the convolution product defining N' by solving a partial differential equation...

As we can not prove constructively conjecture 5, we have no algorithm to test condition 12 that would give a certificate. Therefore, for our implementation, we use a stronger condition using this definition:

Definition 6. *Let $1 < m \leq n$ two natural numbers, let $S \subset \mathrm{Mat}_{m,n}(\mathbb{R})$ a set of matrices. S is said to be* strongly full rank *if*

$$\forall \sigma \in \{-1,1\}^m, 0 \notin \mathcal{H}(\{\sigma A, A \in S\})$$

Proposition 7. *If $S \subset \mathrm{Mat}_{m,n}(\mathbb{R})$ is strongly full rank, then $\mathcal{H}(S)$ contains only full rank matrices.*

Proof. If $A \in \mathcal{H}(S)$ is not full rank, then there exists $v \in \mathbb{R}^m$ such that $vA = 0$. Take $\sigma = (\sigma_1, \dots, \sigma_n)$ such that $\sigma_i = 1$ if $v_i \geq 0$ and $\sigma_i = -1$ otherwise and we find $0 \in \mathcal{H}(\{\sigma A, A \in S\})$. $\square$

This stronger condition gives a weaker conjecture that correspond to our implementation:

Conjecture 8. *Let $(S_i)_{i\in I}$ be a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$, the unit sphere of $\mathbb{R}^{n+1}$. Let $p = (p_1, \dots, p_m)$ be a family of $m \leq n$ positively homogeneous C^1 functionx from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{\ltimes}$ of respective degree $(d_1, \dots, d_m)$ not necessarily equal (i.e. $p_i(\lambda x) = \lambda^{d_i} p_i(x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^{\ltimes+\Bbbk}$). Let $V = \{x \in \mathbb{S}^n, p(x) = 0\}$ be the zero locus of p restricted to $\mathbb{S}^n$.*

We define $\tilde{p} : \mathbb{R}^{n+1} \to \mathbb{R}^m$ the piecewise linear function such that for all $i \in I$, $\tilde{p}|_{S_i}$ is linear and for any v vertex of S_i, we have $p(v) = \tilde{p}|_{S_i}(v)$. We define the following:

- *$\tilde{V} = \{x \in \mathbb{S}^n, \tilde{p}(x) = 0\}$ the zero locus of $\tilde{p}$.*
- *$\mathbb{K}(p) = \{x \in \mathbb{S}^n, \forall i \in \{1, \dots, m\}, p_i(x)\tilde{p}_i(x) \leq 0\}$.*
- *$\tilde{\nabla} p(x) = \{\nabla \tilde{p}|_{S_i}(x), x \in S_i\} \subset \mathrm{Mat}_{m,n+1}(\mathbb{R})$*
- *$G(p,x) = \{\nabla p(x)\} \cup \tilde{\nabla} p(x) \subset \mathrm{Mat}_{m,n+1}(\mathbb{R})$*

If $\forall x \in \mathbb{K}(p)$, $G(p,x)$ is strongly full rank, then V and $\tilde{V}$ are isotopic.

6 Implementation

Obtaining an implementation from theorem 3 or conjecture 8 is not very difficult.

Let $(S_i)_{i\in I}$ be a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$, the unit sphere of $\mathbb{R}^{n+1}$. Let $p = (p_1, \dots, p_m)$ be a family of $m \leq n$ homogeneous polynomials with $n+1$ variables.

Let us consider now a simplicial cone F which is a subset of a face of dimension d_F of one of the simplex S_i for $i \in I$. F could be reduced to a vertex when $d_F = 0$ or could be of dimension $d_F = n + 1$. By doing a change of coordinates sending the unit simplex of dimension d_F to F and writing the resulting polynomials in the Berstein bases, we can use the fact that the value of polynomials or their differentials are in the convex hull of the coefficients to check that the condition of our theorem 3 or conjecture 8 holds in F.

This leads to the following procedure:

Procedure 9 (`TEST_FACE`).
Inputs:

- *$(S_i)_{i\in I}$ a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$.*
- *$p = (p_1, \dots, p_m)$ a family of $m \leq n$ homogeneous polynomials with $n+1$ variables.*
- *$\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_m)$ the piecewise linear functions associated to p and $(S_i)_{i\in I}$.*
- *a simplex F of dimension d_F which is included in a face of one of the S_i.*
- *a minimal size for simplices.*
- *a heuristic to split a simplex in 2 (that may use all other inputs).*

Algorithm:

1. *Build the matrix P sending the unit simplex of dimension d_F to F*
2. *Write $p(M(x))$ and $\tilde{p}(M(x))$ in the Berstein basis, this gives two families of homogeneous polynomials $q(x)$ and $\tilde{q}(x)$ with $d_F + 1$ variables. Remark: if F is a vertex, it is equivalent to evaluating the polynomials!*
3. *If there is $1 \leq i \leq m$ such that all coefficients of q_i and $\tilde{q}_i$ have the same sign, return* `TRUE` *because F does not meet $\mathbb{K}(p)$.*
4. *Otherwise, compute the list L such that $\{S_l, l \in L\}$ is the set all simplicies that contains F.*
5. *Write $\nabla p(M(x))$ and, for all $l \in L$, $\nabla \tilde{p}|_{S_l}(M(x))$ in the Berstein bases. Define the set A of $m \times (n+1)$ matrices which are the coefficients of those polynomials. If for all $\sigma \in \{-1,1\}^m$ we have $0 \notin \mathcal{H}(\{\sigma M, M \in A\})$ return* `TRUE` *because A is strongly full rank.*
6. *Otherwise, if F is not too small, subdivide F in F_1 and F_2 and recursively call the procedure* `TEST_FACE` *on F_1 and F_2 and return* `TRUE` *if both calls return* `TRUE`*.*
7. *Otherwise, if F was too small, return* `FALSE`*.*

Using this procedure, we can implement our main loop:

Procedure 10 (`MAIN_LOOP`)**.**
Inputs:

- *$(S_i)_{i \in I}$ a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones with vertices on $\mathbb{S}^n$.*
- *$p = (p_1, \ldots, p_m)$ a family of $m \leq n$ homogeneous polynomials with $n+1$ variables.*
- *$\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_m)$ the piecewise linear functions associated to p and $(S_i)_{i \in I}$.*
- *a heuristic to refine the decomposition (that may use all other inputs).*
- *a minimal size for simplices.*
- *a heuristic to split a simplex in 2*

Algorithm:

1. *For each face F of one of the simplex call the procedure* `TEST_FACE`*. If all calls return* `TRUE`*, return $\tilde{p}$.*
2. *If the procedure* `TEST_FACE` *returned* `FALSE` *on F, try to refine the decomposition, preferably in a way that splits F.*
3. *Update $\tilde{p}$ to the new decomposition.*
4. *Call back* `MAIN_LOOP` *with the refined subdivition and new $\tilde{p}$.*

Here is the entry point of our implementation:

Procedure 11 (`MAIN`)**.**
Inputs:

- *$p = (p_1, \ldots, p_m)$ a family of $m \leq n$ homogeneous polynomials with $n+1$ variables.*
- *a heuristic to refine a decomposition of $\mathbb{R}^{n+1}$ in simplicial cones.*
- *a minimal size for simplices.*
- *a heuristic to split a simplex in 2.*

Algorithm:

1. *Build* $(S_i)_{i\in I}$ *a decomposition of* $\mathbb{R}^{n+1}$ *in simplicial cones with vertices on* $\mathbb{S}^n$*.*
2. *Build* $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_m)$ *the piecewise linear functions associated to* p *and* $(S_i)_{i\in I}$*.*
3. *Call the procedure* `MAIN_LOOP`*.*
4. *If it return* $\tilde{p}$*, build the piecewise affine projective variety of equation* $\tilde{p}(x) = 0$ *and return it.*

Proposition 12. *Given* $p = (p_1, \ldots, p_m)$ *a family of* $m \leq n$ *homogeneous polynomials with* $n+1$ *variables, if there is only one polynomial or if conjecture 8 is true, the above algorithm loops or returns a piecewise affine projective variety that is isoptopic to the variety defined by* $p(x) = 0$*.*

Proof. This is a consequence of the definitions and the properties of Bernstein basis. It is important to note that the property "0 is the convex hull" used in the algorithm is invariant by a linear change of variable. □

Proposition 13. *Let* $p = (p_1, \ldots, p_m)$ *a family of* $m \leq n$ *homogeneous polynomials with* $n+1$ *variables. Assume that the matrix* $\nabla p(x)$ *is full rank for all* $x \in \mathbb{S}^n$ *such that* $p(x) = 0$*. Then, our algorithm terminates, if the heuristic to refine decompositions, when repeated, gives decompositions such that all simplicial cones have diameter that converges to* 0*, when intersected with the unit sphere.*

Proof. If the diameter of all simplicial cones restricted to $\mathbb{S}^n$ are small enough, then p will be almost linear on each of them and the points where $p(x) = 0$ will be separated from the points where the matrix $\nabla p(x)$ is not full rank. This means that one of the two tests will always succeed in the procedure `TEST_FACE`. □

Test for 0 in the convex hull Clearly we do not want to compute the convex hull to check for one point. This problem is traditionally implemented as a reduction to linear programming. We chose to implement it directly:

For a finite set of vector A, we try to minimise $\|N\|^2$ for $N = \sum_{V\in A} \alpha_V V$ with $\alpha_V > 0$ for all $V \in A$ and $\sum_{V\in A} \alpha_V = 1$. We perform this minimisation by alterning two kinds of steps:

- Linear steps: we solve a linear system to find a direction which is not always a direction of descent but that often offers rapid progress. This kind of steps may set some of the α_V to zero.
- Descent in the direction $V \in A$ if $N.V \leq 0$. It is easy to show that $\|\frac{N+\alpha V}{1+\alpha}\|^2 < \|N\|^2$ in this case. This kind of steps increase α_V, even if $\alpha_V = 0$. We stop if there is no such vector V and we know that $0 \notin \mathcal{H}(A)$.
- We stop if $\|v\|^2$ is too small (meaning we can probably reach 0).
- It is important to avoid setting $\alpha_V = 0$ in a linear step, followed by a descent in the direction V. This yields to very slow progress. Thus, we do not select V for descent if α_V was set to 0 by the previous linear step.

This relatively simple algorithm works very well for this specific case and might be the object of a separate publication in the near future. It is worth noticing that our algorithm is not an interior point method nor a method that stay on the border of the convex hull.

We mentioned the algorithm to make it clear that when we fail to find a descent direction, we have $N.V > 0$ for all $V \in A$ and therefore N is the vector given by the geometric form of Hahn Banach theorem and considered by the proof.

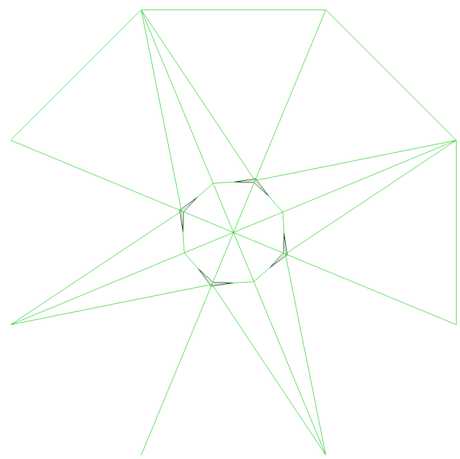

Figure 4: simplicial decomposition for $p_\varepsilon(x, y, t) = (x^2 + y^2 - t^2)^2 + \varepsilon xy(x - y)(x + y)$

Certificate and exact algorithm The above algorithm is implemented using 64 bits floating point numbers. However, the procedure `TEST_FACE` keeps a trace of the subdivision it did as a binary tree and it also keeps in the leaf of the tree a boolean giving the reason of success: either `TRUE` if the sign of the polynomials was constant or `FALSE` if the test for the convex hull succeeded. In the latter case it also keeps the vectors N given by the algorithm for each value of $\sigma \in \{-1, 1\}^m$.

Such a tree is associated to each face of the simplicial decomposition and form a *certificate*.

This allows to recheck the criteria using exact rational arithmetic and the only operations are:

- change of coordinates in the polynomials,
- scalar products and
- comparisons.

As a result the final check of this certificate is fast (in practice faster that the initial computation) and ensures an exact result (if conjecture 8 is true when codimension is greater than 1).

7 Experiments

Note: all figures in this article use a projection of the projective space into a sphere, so we see the entire variety.

Study near singularity Our first example is with the family of quartic polynomials:

$$p_\varepsilon(x, y, t) = (x^2 + y^2 - t^2)^2 + \varepsilon xy(x - y)(x + y)$$

When $\varepsilon > 0$, the curve $p_\varepsilon(x, y, t) = 0$ has four components and it converges to a circle of double points when $\varepsilon \to 0$. However, a simplicial decomposition with only 24 triangles seems sufficient for any $\varepsilon > 0$. Only the number of subdivision of each triangle increases when $\varepsilon \to 0$. Figure 4 is the simplicial decomposition (in green) and the curve $\overline{p}_\varepsilon(x, y, t) = 0$ we get for some $\varepsilon > 0$ (in black):

We now give a table that gives for some value of ε, the total computing time, the time for the exact test using rational arithmetic and the maximum number of time we split a simplex in 2 parts (i.e. maximum depth of recursive call in `TEST_FACE`).

ε	time	$\mathbb{Q}$-time	$\mathbb{Q}$-time/time	max splits
5.10^{-1}	0.079s	0.016s	20%	0
5.10^{-2}	0.150s	0.022s	15%	6
5.10^{-3}	0.264s	0.095s	36%	10
5.10^{-4}	0.607s	0.210s	35%	14
5.10^{-5}	1.638s	0.656s	40%	17
5.10^{-6}	5.380s	2.548s	47%	20
5.10^{-7}	17.087s	6.732s	39%	24

The maximum number of splits seems linear in the exponent of ε, therefore the number of subdivision may be at most linear in ε (some simplices needs less subdivision than others).

We also observe that the final exact test using rational arithmetic never exceeds half of the total running time. Remark: very small ε would require multi-precision which we implemented using GMP. But it is far too slow in practice.

Some curves and surfaces in 2D and 3D It is well know that sextic curves have at most 11 components and that this can be realized in three ways: one oval containing p empty ovals and $10-p$ empty ovals outside for $p = 1, 5$ or 9 [18, 22]. Construction being respectively due to Harnak (figure 5), Hilbert (figure 6) and Gudkov (figure 7). Our implementation succesfully computes the topology of these three curves.

We also experimented succesfullt with two quartic surfaces and two complete intersection of degree 3×2 and 4×2: one maximal quartic (referred as "M quartic", figure 8) with two components, a sphere and a sphere with 10 handles and another quartic with 10 spheres (referred as "M-2 quartic", figure 1 in the introduction). We also show the intersection of four planes (product of four linear forms) and a sphere which are used to build the M-2 quartic (referred as "M-2 quartic $\cap$ S"). Finally, we tested with a 3D curves which is the intersection of a cone and a cubic surface that gives 5 components (referred as "cubic $\cap$ cone"). This last example is used in [12, 13] to construct Del Pezzo surfaces of degree 1.

Results are summarised in the table below where we give for each example the codimension (number of polynomials) and projective dimension (number of variables - 1), the total computing time, the time to check the certificate, the total number of simplices in the decomposition and the maximum number of splits (i.e. maximum depth of recursive call in `TEST_FACE`).

The number of simplices is smaller than the number of simplices in the corresponding Newton polytope (counting all quadrants/octants): $4 \times 36 = 144$ for sextic curves and $8 \times 30 = 240$ for quartic surfaces.

	codim/dim	time	$\mathbb{Q}$-time	simplices	max splits
Harnack's sextic	1/2	1.430s	0.142s	60	8
Gudkov's sextic	1/2	15.748s	1.488s	76	27
Hilbert's sextic	1/2	26.036s	7.724s	84	23
M quartic	1/3	5.243s	0.845s	204	7
M-2 quartic	1/3	4.380s	0.639s	141	7
cubic $\cap$ S	2/3	4.050s	0.280s	54	16
M-2 quartic $\cap$ S	2/3	12.809s	1.304s	87	17

Random varieties We also performed experiments with random Polynomials for Bombieri's norm. This norm is known to give more interesting topology, hence more difficult to compute than with Euclidien norm. It shows the limit of the current implementation: we can compute quartic hyper-surfaces up to dimension 5 in around 20 minutes.

For the zero dimensional case (for which much better approach exists like msolve), we managed to handle in dimension 4 systems with 3 polynomials of degree 2 and one of degree 3 (total degree 24) in around 8 minutes (this takes less than a second with msolve).

You may find in appendix our raw measurements for random polynomials.

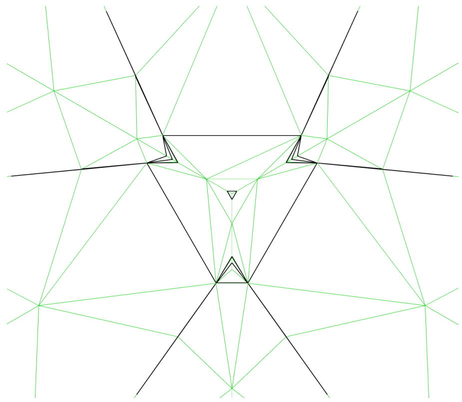

Figure 5: Harnack's sextic

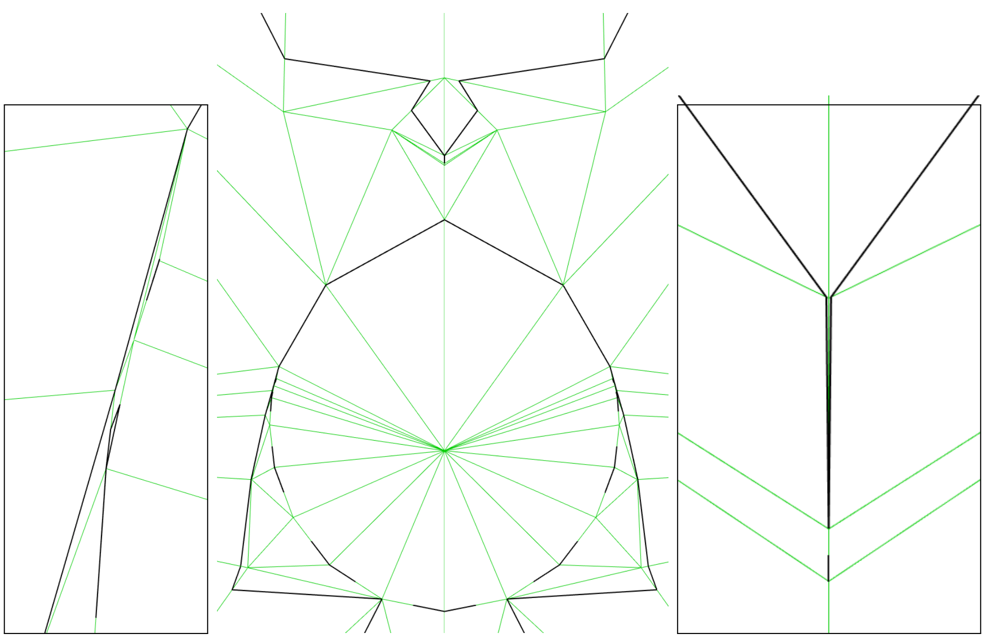

Figure 6: Hilbert's sextix, with two zooms

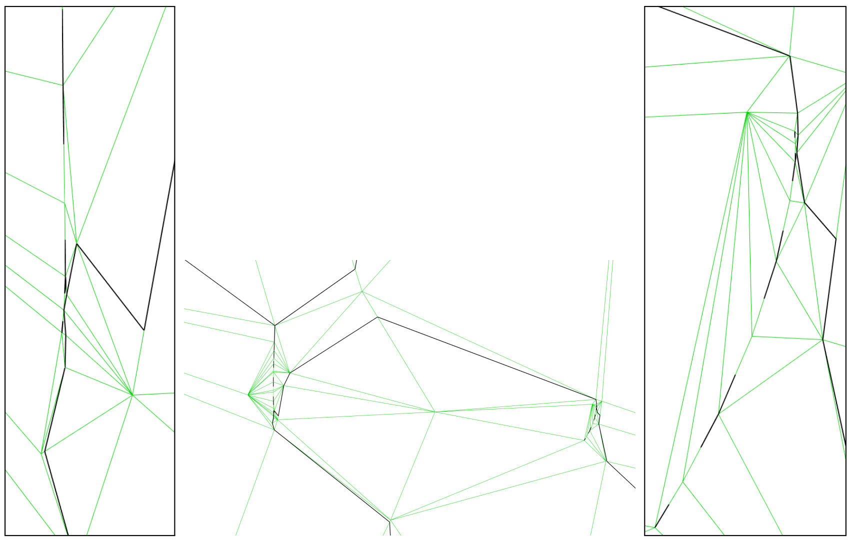

Figure 7: Gudkov's sextix, with two zooms

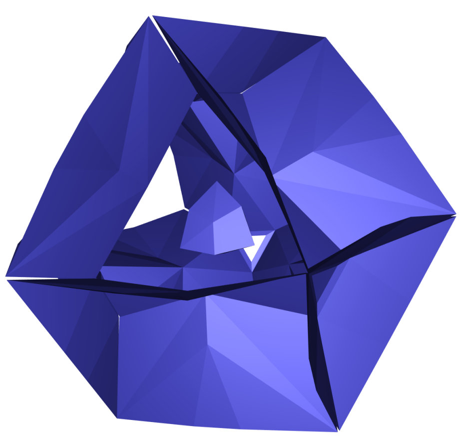

Figure 8: A maximal quartic "M quartic"

It is worth noticing that for all these random tests, the exact test did always succeed. This is not so surprising as random varieties are expected to be *smooth enough*.

Problematic cases As mentioned in the previous section, concentric circles (or near to parallel curves) are currently problematic. We experimented with

$$p(x, y, t) = (x^2 + y^2 - (1 - \alpha)^2)(x^2 + y^2 - (1 + \alpha)^2)$$

We give in the table below the computing time, number of simplices and max splits as above. For a difference of radius of $2\alpha = 10^{-4}$ we need far more simplices than Newton the polytope for a quartic curve $4 \times 16 = 64$. This is rapidely unfeasible.

2α	time	simplices	max splits
1	0.083s	12	1
10^{-1}	0.370s	48	6
10^{-2}	1.793s	128	13
10^{-3}	20.476s	690	20
10^{-4}	417.259s	3504	28

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A Timings on random polynomials

Here is how to read a line in these raw results:

```
4, (2, 2, 2, 3) => 348.0254s [16.8249s] < 364.8503s (2 samples)
```

You can read from left to right:

- the projective dimension,
- the degree of each polynomials,
- the average time to compute a piecewise linear approximation,
- the standard deviation,
- the worst time observed and
- the number of samples.

```
2, (15) => 9.4937s [4.9042s] < 20.0521s (17 samples)
2, (14) => 6.0617s [2.6986s] < 12.4414s (17 samples)
2, (13) => 4.1027s [2.0081s] < 7.2388s (17 samples)
2, (12) => 2.5252s [0.9513s] < 3.8564s (17 samples)
2, (11) => 1.3036s [0.7160s] < 3.0407s (18 samples)
2, (10) => 1.0300s [0.7261s] < 3.2537s (18 samples)
2, (9) => 0.4945s [0.2215s] < 0.8207s (18 samples)
2, (8) => 0.3474s [0.2246s] < 0.9675s (18 samples)
2, (7) => 0.1532s [0.0977s] < 0.4170s (18 samples)
2, (6) => 0.1279s [0.0744s] < 0.2980s (18 samples)
2, (5) => 0.0749s [0.0715s] < 0.2773s (18 samples)
2, (4) => 0.0537s [0.0756s] < 0.3414s (18 samples)
2, (3) => 0.0233s [0.0249s] < 0.0984s (18 samples)
2, (2) => 0.0057s [0.0030s] < 0.0121s (18 samples)
```

```
3, (8) => 64.0352s [12.9702s] < 87.0749s (9 samples)
3, (7) => 17.9244s [4.5345s] < 24.4767s (11 samples)
3, (6) => 6.7120s [2.3389s] < 9.9139s (11 samples)
3, (5) => 2.0908s [0.5444s] < 3.0534s (11 samples)
3, (4) => 0.8662s [0.4510s] < 1.6798s (11 samples)
3, (3) => 0.4839s [0.3853s] < 1.4661s (11 samples)
3, (2) => 0.0368s [0.0165s] < 0.0716s (12 samples)
4, (5) => 91.6522s [43.2871s] < 164.7578s (6 samples)
4, (4) => 21.3854s [9.6121s] < 34.8824s (11 samples)
4, (3) => 3.2500s [1.2441s] < 5.7096s (11 samples)
4, (2) => 0.2308s [0.0108s] < 0.2612s (11 samples)
5, (4) => 1119.8743s [183.6737s] < 1303.5480s (2 samples)
5, (3) => 127.2540s [52.1713s] < 192.3819s (3 samples)
5, (2) => 3.8360s [1.4788s] < 5.0987s (3 samples)
6, (3) => 2361.25s [0] < 2361.25s (1 sample)
6, (2) => 14.4625s [0.1260s] < 14.6000s (5 samples)
2, (8, 8) => 5.2380s [3.2701s] < 12.7953s (9 samples)
2, (7, 8) => 6.2327s [2.8787s] < 10.1879s (9 samples)
2, (7, 7) => 2.5434s [1.5762s] < 6.7730s (15 samples)
2, (6, 8) => 2.5903s [1.8971s] < 6.3826s (9 samples)
2, (6, 7) => 2.7967s [1.2523s] < 5.1338s (15 samples)
2, (6, 6) => 2.1022s [2.2025s] < 8.3182s (16 samples)
2, (5, 8) => 1.9484s [0.8972s] < 3.1652s (10 samples)
2, (5, 7) => 1.6489s [1.2579s] < 5.5694s (16 samples)
2, (5, 6) => 2.3762s [2.2781s] < 7.5396s (16 samples)
2, (5, 5) => 1.1877s [0.9082s] < 3.2313s (16 samples)
2, (4, 8) => 1.3558s [0.7226s] < 2.5465s (10 samples)
2, (4, 7) => 1.1043s [0.5974s] < 2.2448s (17 samples)
2, (4, 6) => 0.8182s [0.5225s] < 1.7687s (17 samples)
2, (4, 5) => 0.6955s [0.5440s] < 2.1073s (17 samples)
2, (4, 4) => 0.7081s [0.7159s] < 2.6433s (17 samples)
2, (3, 8) => 1.1122s [0.6825s] < 2.6819s (11 samples)
2, (3, 7) => 1.0376s [0.7008s] < 3.1081s (17 samples)
2, (3, 6) => 0.7728s [0.7388s] < 3.3402s (17 samples)
2, (3, 5) => 0.4749s [0.3775s] < 1.3720s (17 samples)
2, (3, 4) => 0.3867s [0.3490s] < 1.3088s (17 samples)
2, (3, 3) => 0.3086s [0.3875s] < 1.6391s (17 samples)
2, (2, 8) => 0.5736s [0.4633s] < 1.7001s (11 samples)
2, (2, 7) => 0.8804s [1.4395s] < 4.9613s (17 samples)
2, (2, 6) => 0.2350s [0.1518s] < 0.6401s (17 samples)
2, (2, 5) => 0.2294s [0.2023s] < 0.8916s (17 samples)
2, (2, 4) => 0.3407s [0.6465s] < 2.8759s (17 samples)
2, (2, 3) => 0.0939s [0.1056s] < 0.4515s (17 samples)
2, (2, 2) => 0.0578s [0.0621s] < 0.2712s (17 samples)
3, (5, 5) => 88.8346s [39.5227s] < 157.2866s (8 samples)
3, (4, 5) => 43.8346s [13.6107s] < 64.2336s (8 samples)
3, (4, 4) => 23.7975s [14.3489s] < 61.8225s (9 samples)
3, (3, 5) => 28.8840s [25.3741s] < 94.8858s (9 samples)
3, (3, 4) => 13.9328s [6.0994s] < 23.6953s (9 samples)
3, (3, 3) => 6.5742s [2.0125s] < 9.1454s (9 samples)
3, (2, 5) => 16.0180s [13.6073s] < 52.4697s (9 samples)
3, (2, 4) => 11.2471s [12.0677s] < 40.5447s (9 samples)
3, (2, 3) => 1.7853s [1.1207s] < 3.9345s (9 samples)
3, (2, 2) => 0.9715s [1.2720s] < 4.2365s (9 samples)
4, (2, 3) => 120.8822s [43.8194s] < 204.6149s (6 samples)
4, (2, 2) => 52.8344s [44.7051s] < 123.6788s (8 samples)
3, (4, 4, 4) => 57.0216s [24.2671s] < 102.3699s (8 samples)
3, (3, 4, 4) => 63.4938s [22.6517s] < 107.4020s (8 samples)
3, (3, 3, 4) => 39.8249s [18.1909s] < 72.8629s (8 samples)
3, (3, 3, 3) => 21.1542s [13.3550s] < 50.7229s (8 samples)
3, (2, 4, 4) => 52.3019s [55.8282s] < 196.6823s (8 samples)
3, (2, 3, 4) => 14.7853s [9.9172s] < 31.4450s (8 samples)
3, (2, 3, 3) => 11.2110s [4.0091s] < 18.0787s (8 samples)
3, (2, 2, 4) => 14.8802s [9.2282s] < 38.0585s (8 samples)
3, (2, 2, 3) => 5.7705s [2.6911s] < 9.0264s (8 samples)
3, (2, 2, 2) => 5.9955s [4.1095s] < 14.1836s (8 samples)
4, (2, 2, 3) => 480.8930s [111.0565s] < 573.5631s (3 samples)
4, (2, 2, 2) => 166.5942s [86.3898s] < 292.1297s (4 samples)
4, (2, 2, 2, 3) => 348.0254s [16.8249s] < 364.8503s (2 samples)
4, (2, 2, 2, 2) => 247.7984s [24.4447s] < 282.2358s (3 samples)
```