EXISTENCE AND UNIQUENESS OF SOLUTIONS OF ODE WITH DISCONTINUITY AT $t = 0$

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Abstract. We give an existence and uniqueness result of solutions of ordinary differential equations $x' = f(t, x)$ when f has discontinuity at $t = 0$, and the time dependent Lipschitz constant $L(t)$ for $x \mapsto f(t, x)$ is not integrable near 0.

1. Motivation

We start with an example: we consider the 2 differential equations:

$$x' = \tanh\left(\frac{x}{|t\ln(|t|)|}\right) \qquad\qquad x' = \tanh\left(\frac{x}{|t|}\right)$$

None of them are Lipschitz when t converges to 0. Nevertheless, as shown in figure 1, solving these equations numerically[1] seems to indicate continuity of the solutions in the initial conditions for the first one, while this seems not to be the case for the second one. In that latter case, the gap of ~ 0.15 observed at $t = 0.5$ corresponds to a difference of 2^{-50} in the initial condition at $t = -0.1$.

Note: To have stable numerical result, especially in the second example, we had to use very small steps near $t = 0$.

Here is a definition that captures these examples:

Definition 1. *Let $I =]-1, 1[$, let $\varphi : I \to \mathbb{R}_+^*$, $\varphi(t) = |t\ln(|t|)|$, let $X^\star$ be $I \setminus \{0\} \times \mathbb{R}^n$. We say the $f : (t, x) \in X^\star \to \mathbb{R}^n$ is φ-Lipschitz, if*

(1) f is continuous on $X^\star$.
(2) $\forall (t, x) \in X^\star, \|f(t, x)\| \leq K(|ln(|t|)| + 1)$ for some $K \in \mathbb{R}$.
(3) $\exists L \in \mathbb{R}_+^*, \forall x, y \in \mathbb{R}^n, (t, x) \in X^\star, (t, y) \in X^\star$ imply

$$\|f(t, x) - f(t, y)\| \leq L\varphi(t)\|x - y\|$$

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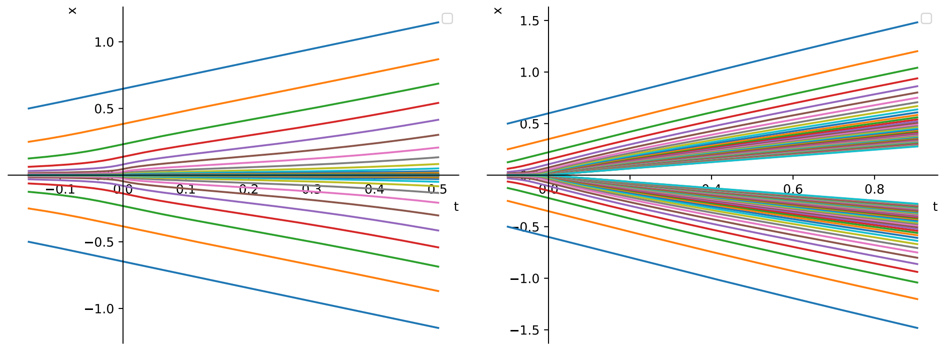
[1]The python code of the examples, using numpy, is available from the web page of the author at `https://raffalli.eu/downloads/rk4.py`.



$x' = \tanh\left(\frac{x}{|t\ln(|t|)|}\right)$ $x' = \tanh\left(\frac{x}{|t|}\right)$

FIGURE 1. Comparison of solutions of two ODEs with initial condition $x(-0.1) = \pm 2^{-i}$ for $i = 1, \dots, 50$. It is computed using RK4 method with $\mathrm{d}t = \max(10^{-2}t, 10^{-14})$

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The first example is φ-Lipschitz, while the second if not. The main result of the paper (theorem 6) is to prove existence and uniqueness of solutions for O.D.E. with $x'(t) = f(t, x(t))$ when f is φ-Lipschitz.

Among the many existence and uniqueness theorems for ODE ([7, 3, 2, 6, 5, 9], there are specific results where the Lipschitz condition is time dependent. There is a result by Hartman and Wintner [4] that would give existence and uniqueness if φ had a finite integral or a result by Osgood [8] if $\frac{1}{\varphi}$ had an infinite integral, but these do not apply. There seems to be no result covering the first example above or our definition. For the sole existence of solutions, Carethéodory's theorem [1] allows to conclude, but we choose to reprove existence for the sake of completeness.

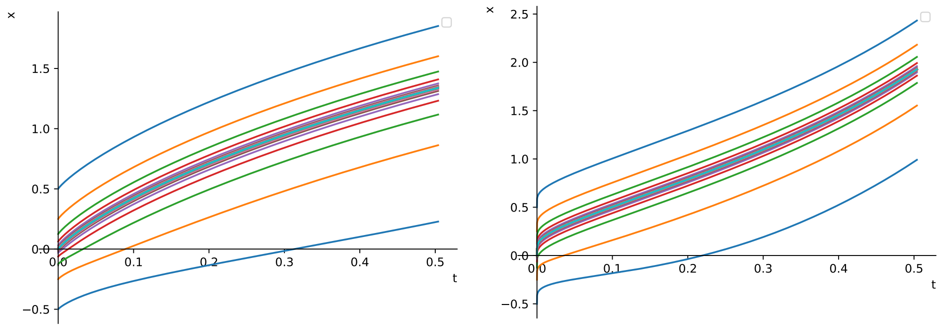
The condition $f(t, x) \leq K|\ln(|t|)| + 1$ is essential to our proof, but we conjecture[2] that it may be relaxed by $f(t, x) \leq m(|t|)$ with $m(t)$ being integral as suggested by the second example of figure 2:

2. Some useful functions

Definition 2. *Let $L \in \mathbb{R}_+^*$, we define (see figure 3)*

- $\varphi :]-1, 1[\setminus\{0\} \to \mathbb{R}_+^*$, $\varphi(t) = \frac{1}{|t\ln(|t|)|}$,
- $\Psi_L :]-1, 1[\to \mathbb{R}_+^*$, $\Psi_L(0) = 0, \Psi_L(t) = (-\ln(|t|))^{-L}$ *if* $t \neq 0$.

[2]If a reader has a proof covering the case where f is only integrable, let the author of this note know, I have very interesting applications...



$x' = |\ln(|t|)| + \tanh\left(\frac{x}{|t\ln(|t|)|}\right)$ $\quad$ $x' = \frac{1}{|t\ln^2(|t|)|} + \tanh\left(\frac{x}{|t\ln(|t|)|}\right)$

FIGURE 2. Examples with f unbounded, our theorem applies to the first example, still the second example seems to admit unique solutions

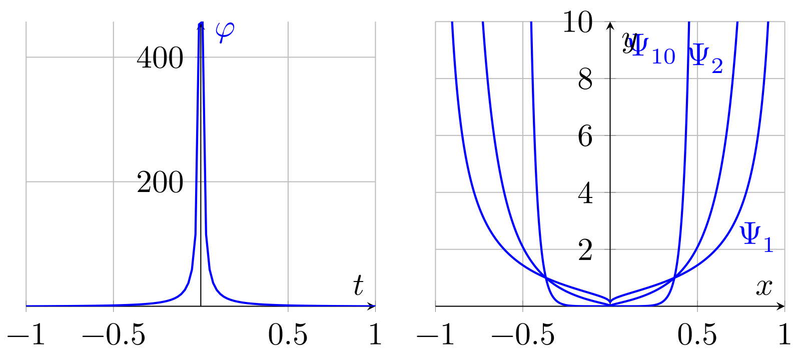


FIGURE 3. Plot of φ and Ψ_L

Lemma 3. *We have for* $t \in]0,1[$, $\Psi'(t) = L\varphi(t)\Psi(t)$ *and for* $t \in]-1,0[$, $\Psi'(t) = -L\varphi(t)\Psi(t)$.

Proof. If $t > 0$,
$$\begin{aligned}\Psi_L'(t) &= -L(-t)^{-1}(-\ln(t))^{-L-1}\\ &= Lt^{-1}(-\ln(t))^{-1}\Psi_L(t)\\ &= L\varphi(t)\Psi_L(t) \text{ because } \varphi(t) = \frac{1}{-t\ln(t)} \text{ for } t > 0.\end{aligned}$$
If $t < 0$,
$$\begin{aligned}\Psi_L'(t) &= -Lt^{-1}(-\ln(-t))^{-L-1}\\ &= -Lt^{-1}(-\ln(-t))^{-1}\Psi_L(t)\\ &= L\varphi(t)\Psi_L(t) \text{ because } \varphi(t) = \frac{1}{t\ln(-t)} \text{ for } t < 0.\end{aligned}$$
□

Definition 4. *we define* $g : [0,1] \to [0,2]$, $g(t) = -t\ln(t) + 2t$ *and* $h(t) = g^{-1}(t)$ *which exists as* g *is non decreasing. We have* $g'(t) = -\ln(t) + 1$ *which is our bound for* f*.*

Lemma 5. *For* $L > 0$, $\lim_{t\to 0} t\ln(h(t))^L = 0$.

Proof. We have $\lim_{t\to 0} g(t) = 0$, so we can do the change of variable $u = h(t)$ hence $t = g(u)$ and we find $t\ln(h(t))^L = g(u)\ln(u)^L = (-u\ln(u) + 2u)\ln(u)^L$ whose limit is indeed 0. □

3. Our main result

Lemma 6. *Using the notations of definition 1, if* $f : (t,x) \in X^\star \to \mathbb{R}^n$ *is* φ*-Lipschitz then, the differential equation*

$$x'(t) \quad = \quad f(t,x(t)) \tag{1}$$

admits solutions with the initial condition $x(t_0) = x_0$ *for* $(t, x_0) \in X$ *(even for* $t_0 = 0$*). Furthermore, the solutions are continuous with respect to the initial conditions, hence unique.*

Proof. It is clear that f is Lipschitz in the neighbourhood of any $t_0 \in I\setminus\{0\}$, hence by the standard Picard–Lindelöf-Cauchy-Lipschitz theorem the equation (1) admit solutions for any initial condition $x(t_0) = x_0 \in \mathbb{R}^n$. We now show the existence of local solutions for initial conditions $x(0) = x_0 \in \mathbb{R}^n$ with $(0, x_0) \in X$.

Let us define $b = \min(1, \frac{1}{2L})$ and the interval $J =]-b, b[\subset I$. We define Ψ the set of continuous $\gamma : J \to \mathbb{R}^n$ satisfying:

- γ is C^1 on $J \setminus \{0\}$,
- $\gamma(0) = x_0$,
- and such that $\|\gamma - x_0\|_\varphi \leq K$ where $\|.\|_\varphi$ is

$$\|\gamma\|_\varphi = \max_{t\in J\setminus\{0\}} \varphi(t)\|\gamma(t)\|$$

We define the following operator on Ψ:

$$F(x)(t) = x_0 + \int_0^t f(u, x(u))\mathrm{d}u$$

We first show that $F(x) \in \Psi$. It is clear that $F(x)$ is of class C^1 on $J \setminus \{0\}$, as f is continuous. It is also clear that $F(x)(0) = x_0$. Therefore $F(x)$ is defined on J. Finally, we have:

$$\varphi(t)\|F(x)(t) - x_0\| \quad \leq \quad \varphi(t)\left|\int_0^t \|f(u, x(u))\|\mathrm{d}u\right|$$

$$\begin{aligned}
&\le \varphi(t)\left|\int_0^t K(|\ln|t||+1)\right| \\
&\le \varphi(t)K(|t\ln|t||+2|t|) \\
&\le K+2K|\ln|t||^{-1} \\
&\le 3K
\end{aligned}$$

This proves that $F:\Psi\to\Psi$. Now, we prove that F is contracting for the norm $\|\_\|_\varphi$: for any $t\in J$ and $x,y\in\Psi$, we have:

$$\begin{aligned}
\|F(x)(t)-F(y)(t)\| &\le \left|\int_0^t \|f(u,x(u))-f(u,y(u))\|\mathrm{d}u\right| \\
&\le \left|\int_0^t L\varphi(t)\|x(u)-y(u)\|\mathrm{d}u\right| \\
&\le L\left|\int_0^t \mathrm{d}u\right|\|x-y\|_\varphi \\
&\le Lb\|x-y\|_\varphi \\
&\le \frac{1}{2}\|x-y\|_\varphi \text{ because } b<\frac{1}{2L}
\end{aligned}$$

Therefore, F is contracting on Ψ for the norm $\|\_\|_\varphi$. This ensures the existence of a fixpoint of F, i.e. the existence of a solution in J with initial condition $x(0)=x_0$.

Note: the above proof does not establish uniqueness, or more precisely, it establish uniqueness of local solutions such that $\varphi(t)\|F(x)(t)-x_0\|$ is bounded We need the following result to get both uniqueness and continuity of arbitrary solutions with respect to the initial condition:

Lemma 7. *Using the notations of the previous definition, if $f:(t,x)\in X^\star\to\mathbb{R}^n$ is φ-Lipschitz then, if x and y are two solutions defined on $[u,t]$ (resp. $[-u,-t]$) where $0\le u,t<1$, then we have*

$$\|y(t)-x(t)\|\le(2K+1)\left|\frac{\ln(\max(u,\min(t,h(\|y(u)-x(u)\|))))}{\ln(t)}\right|^L\|y(u)-x(u)\|$$

See definition 4 for the definition of h.

This implies continuity of solutions with respect to the initial condition. In $u=0$, this uses lemme 5 to have $\lim_{v\to0}\ln(h(|v|))^Lv=0$. To get continuity when t and u have opposite signs, we use the above result twice with $0,t$ and $0,u$. This ends the proof of the main theorem. □

Proof of the above lemma. Let us consider two solutions x and y defined on $J=[u,t]$ with $0\le u<t\in I$. In what follows, we will write $\overline{u}=\max(u,\min(t,h(\|y(u)-x(u)\|)))$. If $\|y(u)-x(u)\|>1$, we consider that h is infinite and $\overline{u}=u$.

We have $u \leq \overline{u} \leq t$ and we compute:

$$\begin{array}{rcl}
y(t) - x(t) & = & y(u) - x(u) + \displaystyle\int_u^t f((s, y(s)) - f(s, x(s)))ds \\
\|y(t) - x(t)\| & \leq & \|y(u) - x(u)\| + \displaystyle\int_u^t \|f((s, y(s)) - f(s, x(s)))\|ds \\
\|y(t) - x(t)\| & \leq & |y(u) - x(u)\| + \\
& & 2K \displaystyle\int_u^{\overline{u}} (-\ln(s) + 1)ds + \\
& & \displaystyle\int_{\overline{u}}^t \|f((s, y(s)) - f(s, x(s)))\|ds \\
\|y(t) - x(t)\| & \leq & \|y(u) - x(u)\| + 2K(g(\overline{u}) - g(u)) + \\
& & \displaystyle\int_{\overline{u}}^t L\varphi(s)\|y(s)) - x(s)\|ds \\
\|y(t) - x(t)\| & \leq & (2K + 1)\|y(u) - x(u)\| + \\
& & \displaystyle\int_{\overline{u}}^t L\varphi(s)\|y(s)) - x(s)\|ds \qquad (2)
\end{array}$$

Remark: if $\overline{u} \leq u \leq t$, the $2K$ term is useless. We define for $\overline{u} \leq v \leq t$:

$$F(v) = \int_{\overline{u}}^v L\varphi(s)\|y(s)) - x(s)\|ds$$

We have $F'(v) = L\varphi(v)\|y(v)) - x(v)\|$ and therefore by (2):

$$\|y(v) - x(v)\| - F(v) \leq (2K + 1)\|y(u) - x(u)\|$$

Multiplying both sides by $\Psi'_L(v)$ gives

$$\Psi'_L(v)\|y(v) - x(v)\| - \Psi'_L(v)F(v) \leq (2K + 1)\Psi'_L(v)\|y(u) - x(u)\|$$

Using lemma 3:

$$\Psi_L(v)F'(v) - \Psi'_L(v)F(v) \leq (2K + 1)\Psi'_L(v)\|y(u) - x(u)\|$$

Multiplying both sides by $\Psi_L^{-2}(v)$ gives

$$\frac{\partial}{\partial v}(\Psi_L(v)^{-1}F(v)) \leq (2K + 1)\Psi'_L(v)\Psi_L(v)^{-2}\|y(u) - x(u)\|$$

Integrating between $\overline{u}$ and t, using $F(\overline{u}) = 0$:

$$\Psi_L(t)^{-1}F(t) \leq (2K + 1)(-\Psi_L(t)^{-1} + \Psi_L(\overline{u})^{-1})\|y(u) - x(u)\|$$

$$F(t) \leq (2K + 1)\left(\frac{\Psi_L(t)}{\Psi_L(\overline{u})} - 1\right)\|y(u) - x(u)\|$$

Using (2):

$$\|y(t) - x(t)\| \leq (2K+1)\frac{\Psi_L(t)}{\Psi_L(\overline{u})}\|y(u) - x(u)\|$$

Replacing Ψ_L by its definition, we find for $0 \leq u < t < 1$:

$$\|y(t) - x(t)\| \leq (2K+1)\left|\frac{\ln(\overline{u})}{\ln(t)}\right|^L \|y(u) - x(u)\|$$

This is the wanted inequality when $u < t$. We now search an inequality in the opposite direction. Similarly to the previous case, we write $\overline{u} = \max(u, \min(t, \|y(t) - x(t)\|))$ (notice the change, as t replaces u). We have

$$\begin{aligned}
y(t) - x(t) &= y(u) - x(u) + \int_u^t f((s, y(s)) - f(s, x(s)))ds \\
\|y(u) - x(u)\| &\leq \|y(t) - x(t)\| + \int_u^t \|f((s, y(s)) - f(s, x(s)))\|ds \\
\|y(u) - x(u)\| &\leq \|y(t) - x(t)\| + \\
&\quad 2K\int_u^{\overline{u}} (-\ln(s) + 1)ds + \\
&\quad \int_{\overline{u}}^t \|f((s, y(s)) - f(s, x(s)))\|ds \\
\|y(u) - x(u)\| &\leq \|y(t) - x(t)\| + 2K(g(\overline{u}) - g(u)) + \\
&\quad \int_{\overline{u}}^t L\varphi(s)\|y(s)) - x(s)\|ds \\
\|y(t) - x(t)\| &\leq (2K+1)\|y(u) - x(u)\| + \\
&\quad \int_{\overline{u}}^t L\varphi(s)\|y(s)) - x(s)\|ds \qquad (3)
\end{aligned}$$

We define

$$F(v) = \int_v^t L\varphi(s)\|y(s)) - x(s)\|ds$$

We have $F'(v) = -L\varphi(v)\|y(v)) - x(v)\|$ and therefore by (3), we have

$$\|y(v) - x(v)\| - F(v) \leq (2K+1)\|y(t) - x(t)\|$$

Multiplying both sides by $\Psi'_L(v) > 0$

$$\Psi'_L(v)\|y(v) - x(v)\| - \Psi'_L(v)F(v) \leq (2K+1)\Psi'_L(v)\|y(t) - x(t)\|$$

Using lemma 3:

$$-\Psi_L(v)F'(v) - \Psi'_L(v)F(v) \leq (2K+1)\Psi'_L(v)\|y(t) - x(t)\|$$

$$-\frac{\partial}{\partial v}(\Psi_L(v)F(v)) \leq (2K+1)\Psi'_L(v)\|y(t)-x(t)\|$$

Integrating for v between $\overline{u}$ and t, using $F(t)=0$

$$\Psi_L(\overline{u})F(\overline{u}) \leq (2K+1)(\Psi_L(t)-\Psi_L(\overline{u}))\|y(t)-x(t)\|$$

$$F(\overline{u}) \leq (2K+1)\left(\frac{\Psi_L(t)}{\Psi_L(\overline{u})}-1\right)\|y(t)-x(t)\|$$

Using (3)

$$\|y(u)-x(u)\| \leq (2K+1)\frac{\Psi_L(t)}{\Psi_L(\overline{u})}\|y(t)-x(t)\|$$

Replacing Ψ_L by its definition, we find for $0 \leq u < t < 1$:

$$\|y(u)-x(u)\| \leq (2K+1)\left|\frac{\ln(\overline{u})}{\ln(t)}\right|^L \|y(t)-x(t)\|$$

It remains to show the same equations for $1 < t, u \leq 0$: this can be deduced from the positive case by the change of variable $t \mapsto -t$. □

References

[1] Constantin Carathéodory. Untersuchungen über die grundlagen der variationsrechnung. *Mathematische Annalen*, 76:369–404, 1918.

[2] Augustin-Louis Cauchy. *Cours d'analyse de l'École Royale Polytechnique*. Imprimérie Royale, Paris, 1821.

[3] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers, 1988.

[4] Philip Hartman and Aurel Wintner. Ordinary differential equations. *The American Mathematical Monthly*, 62(5):250–260, 1955.

[5] Ernst Leonard Lindelöf. Sur l'application de la méthode des approximations successives aux équations différentielles ordinaires du premier ordre. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 8(1):177–185, 1894.

[6] Rudolf Lipschitz. Zur integration der differentialgleichungen. *Journal für die reine und angewandte Mathematik*, 1875:52–70, 1875.

[7] Hiroshi Okamura. Existence and uniqueness of solutions of ordinary differential equations. *Osaka Mathematical Journal*, 9(1):23–38, 1957.

[8] William Fogg Osgood. Beweis der existenz einer lösung der differentialgleichung $\frac{dy}{dx} = f(x,y)$ ohne hinzunahme der cauchy-lipschitz'schen bedingung. *Monatshefte für Mathematik und Physik*, 9(1):331–345, 1898.

[9] Émile Picard. Sur l'approximation des solutions des équations différentielles et sur les solutions des équations aux dérivées partielles. *Journal de Mathématiques Pures et Appliquées*, 6:145–210, 1890.